4.19 To develop a transfer function of a nonlinear process model, (1) linearize the nonlinear model about a steady-state operating point, (2) convert to deviation variables, (3) take the Laplace transform of the equation, and (4) algebraically solve for the transfer function of interest.

4.20 First-order dynamics results for linearization of a nonlinear function using a Taylor series expansion after truncating the quadratic and higher terms.

Analytical Questions and Exercises

4.21 (a) $F(s)=\frac{1}{s+a}+\frac{\omega}{(s+a)^2+\omega^2}$

(b) $F(s)=sG(s)-g(0)-\frac{s+2}{(s+2)^2+25}$

(c) $F(s)=\frac{G(s)}{s}-\frac{6}{(s+5)^4}$

(d) $F(s)=7e^{-10s}G(s)+\frac{10}{s^2+100}$

4.22 (a) Initial value theorem $Lim_{s\longrightarrow\infty}\left[\frac{\omega s}{(s+a)^2+\omega^2}\right]=0$

Final value theorem $Lim_{s\longrightarrow 0}\left[\frac{s\omega}{(s+a)^2+\omega^2}\right]=0$

(b) Initial value theorem $Lim_{s\longrightarrow\infty}\left[\frac{2s}{(s+5)^3}+\frac{5s}{s^2+25}\right]=0$

Final value theorem $Lim_{s\longrightarrow 0}\left[\frac{2s}{(s+5)^3}+\frac{5s}{s^2+25}\right]=0$

(c) Initial value theorem $Lim_{s\longrightarrow\infty}\left[\frac{2s}{(s+2)^2+4}-\frac{s}{s+2}\right]=-1$

Final value theorem $Lim_{s\longrightarrow 0}\left[\frac{2s}{(s+2)^2+4}-\frac{s}{s+2}\right]=0$

(d) Initial value theorem $Lim_{s \longrightarrow \infty}\left[\dfrac{\omega s}{s^2+\omega^2}+\dfrac{s^2}{s^2+\omega^2}\right]=1$

Final value theorem $Lim_{s \longrightarrow 0}\left[\dfrac{\omega s}{s^2+\omega^2}+\dfrac{s^2}{s^2+\omega^2}\right]=0$

4.23 (a)

$$\frac{2}{(s+2)(s+3)}=\frac{A}{s+2}+\frac{B}{s+3}=\frac{A(s+3)+B(s+2)}{(s+2)(s+3)}$$
$$A=2;\ B=-2$$
$$\frac{2}{(s+2)(s+3)}=\frac{2}{s+2}-\frac{2}{s+3}$$

(b)

$$\frac{2}{s^2+11s+30}=\frac{2}{(s+5)(s+6)}=\frac{A}{s+5}+\frac{B}{s+6}=\frac{A(s+5)+B(s+6)}{(s+5)(s+6)}$$
$$A=2;\ B=-2$$
$$\frac{2}{s^2+11s+30}=\frac{2}{s+5}-\frac{2}{s+6}$$

(c)

$$\frac{7}{(s+1)(s+2)(s+6)}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+6}$$
$$A=7/5;\ B=-7/4;\ C=7/20$$
$$\frac{7}{(s+1)(s+2)(s+6)}=\frac{7/5}{s+1}-\frac{7/4}{s+2}+\frac{7/20}{s+6}$$

(d)

$$\frac{3s}{(s+1)(s+7)} = \frac{A}{s+1} + \frac{B}{s+7} = \frac{A(s+7)+B(s+1)}{(s+1)(s+7)}$$

$A = -1/2;\ B = 7/2$

$$\frac{3s}{(s+1)(s+7)} = -\frac{1/2}{s+1} + \frac{7/2}{s+7}$$

4.24 **(a)** $Y(s) = \frac{-1}{s+2} + \frac{2}{s+3}$ by partial fraction expansion

Applying inverse Laplace transforms to each term yields:

$$y(t) = -e^{-2t} + 2e^{-3t}$$

(b) Applying a partial fraction expansion yields:

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Applying an inverse Laplace transform to each term in this equation yields:

$$y(t) = e^{-t} - e^{-2t}$$

(c) Applying a partial fraction expansion yields:

$$Y(s) = \frac{1}{s+1} + \frac{2}{(s+1)^2}$$

Applying an inverse Laplace transform to each term in this equation yields:

$$y(t) = e^{-t} + 2te^{-t}$$

(d) Applying a partial fraction expansion yields:

$$Y(s) = \frac{1}{30(s+1)} + \frac{4}{5(s+6)} - \frac{5}{6(s+7)}$$

Applying an inverse Laplace transform to each term in this equation yields:

$$y(t) = \frac{1}{30}e^{-t} + \frac{4}{5}e^{-6t} - \frac{5}{6}e^{-7t}$$

4.25 Remember in order to apply the final and initial valued theorems the transfer function is first multiplied by *s* and then the limit of *s* approaching zero (final-value theorem) or infinity (initial-value theorem).

(a) $Lim\left(\frac{s(s+1)}{(s+2)(s+3)}\right) = 0\ (s \to 0);\ =1\ (s \to \infty)$ final and initial value, respectively which agrees with the time domain solutions.

(b) $Lim\left(\frac{s}{(s+1)(s+2)}\right) = 0(s \to 0);\ =0(s \to \infty)$ final and initial value, respectively which agrees with the time domain solutions.

(c) $Lim\left(\frac{s(s+3)}{(s+1)^2}\right) = 0(s \to 0);\ =1(s \to \infty)$ final and initial value, respectively which agrees with the time domain solutions.

(d) $Lim\left(\frac{s(s+2)}{(s+1)(s+6)(s+7)}\right) = 0(s \to 0);\ =0(s \to \infty)$ final and initial value, respectively which agrees with the time domain solutions.

4.26 Applying Laplace transforms to each term in the equation yields

$$sY(s) - y(0) = \frac{2}{s^3}$$

Solving for *Y*(*s*) yields $Y(s) = \frac{2}{s^4}$

Applying inverse Laplace transforms to each term yields

$$y(t) = \frac{1}{3}t^3$$

4.27 In each case, Laplace transforms are applied to each term in the differential equation, *Y*(*s*) is solved, and the inverse Laplace transform is applied to yield the time domain solution.

(a) Applying Laplace transform to the ODE $s^2\,Y(s) + 5s\,Y(s) + 6\,Y(s) = \frac{2}{s}$

Solving for *Y*(*s*), factoring, and applying a partial fraction expansion yields

$$Y(s)=\frac{2}{s(s+2)(s+3)}=\frac{1}{3s}-\frac{1}{s+2}+\frac{2}{3(s+3)}$$

Applying inverse Laplace transforms yields the time series solution

$$y(t)=\frac{1}{3}\ -e^{-2t}+\frac{2}{3}e^{-3t}$$

which can be seen to be stable since $y(t)$ is bounded as t approaches infinity.

(b) Applying Laplace transform to the ODE $\quad s^2\,Y(s)+3s\,Y(s)\ =\frac{5}{s}$

Solving for $Y(s)$, factoring, and applying a partial fraction expansion yields

$$Y(s)=\frac{5}{s^2\,(s+3)}=-\frac{5}{9s}+\frac{5}{3s^2}+\frac{5}{9(s+3)}$$

Applying inverse Laplace transforms yields the time series solution

$$y(t)=-\frac{5}{9}+\frac{5t}{3}+\frac{5}{9}e^{-3t}$$

which is unstable due to the linear term in t.

(c) Applying Laplace transform to the ODE $\quad s^2\,Y(s)+4s\,Y(s)-5Y(s)=\frac{4}{s}$

Solving for $Y(s)$, factoring, and applying a partial fraction expansion yields

$$Y(s)=\frac{4}{s(s+5)(s-1)}=-\frac{4}{5s}+\frac{1}{15(s+5)}+\frac{2}{3(s-1)}$$

Applying inverse Laplace transforms yields the time series solution

$$y(t)=-\frac{4}{5}+\frac{2}{15}e^{-5t}+\frac{2}{3}e^{t}$$

which is unstable due to the e^t term.

(d) Applying Laplace transform to the ODE $\quad s^2\,Y(s)+4s\,Y(s)+4Y(s)=\frac{1}{s}$

Solving for $Y(s)$, factoring, and applying a partial fraction expansion yields

$$Y(s)=\frac{1}{s(s+2)^2}=\frac{1}{4s}-\frac{1}{4(s+2)}-\frac{1}{2(s+2)^2}$$

Applying inverse Laplace transforms yields the time series solution

$$y(t)=\frac{1}{4}-\frac{1}{4}e^{-2t}-\frac{1}{2}t\,e^{-2t}$$

which can be seen to be stable since $y(t)$ is bounded as t approaches infinity.

(e) Applying Laplace transform to the ODE $2s^2\,Y(s)+11s\,Y(s)+12Y(s)=\frac{5}{s}$

Solving for $Y(s)$, factoring, and applying a partial fraction expansion yields

$$Y(s)=\frac{5}{s(2s+3)(s+4)}=\frac{5/12}{s}-\frac{2/3}{s+3/2}-\frac{1/4}{s+4}$$

Applying inverse Laplace transforms yields the time series solution

$$y(t)=\frac{5}{12}-\frac{2}{3}e^{-3/2t}+\frac{1}{4}e^{-4t}$$

which can be seen to be stable since $y(t)$ is bounded as t approaches infinity.

(f) Applying Laplace transform to the ODE $3s^2\,Y(s)+7s\,Y(s)+2Y(s)=\frac{6}{s^2}$

Solving for $Y(s)$, factoring, and applying a partial fraction expansion yields

$$Y(s)=\frac{6}{s^2\,(3s+1)(s+2)}=-\frac{21/2}{s}+\frac{3}{s^2}+\frac{54/5}{s+1/3}-\frac{3/10}{s+2}$$

Applying inverse Laplace transforms yields the time series solution

$$y(t)=-\frac{21}{2}+3t+\frac{54}{5}e^{-1/3t}-\frac{3}{10}e^{-2t}$$

which can be seen to be stable since $y(t)$ is bounded as t approaches infinity.

4.28 Taking the Laplace transform of each term in both equations yields

$$\begin{aligned}(s+2)Y_1(s) + \quad Y_2(s) &= \frac{2}{s}\\ Y_1(s) + (s+1)Y_2(s) &= 0\end{aligned}$$

Algebraically solving for $Y_1(s)$ and $Y_2(s)$ results in

$$Y_1(s)=\frac{2(s+1)}{s(s^2+3s+1)}$$

$$Y_2(s)=\frac{-2}{s(s^2+3s+1)}$$

Applying partial fraction expansions and inverse Laplace transforms yields **after considerable algebra!**

$$y_1(t)=\frac{2}{ab}+\frac{2(1+a)}{a(a-b)}e^{-at}+\frac{2(1+b)}{b(b-a)}e^{-bt}$$

$$a=\left(-\frac{3}{2}+\frac{\sqrt{5}}{2}\right) \qquad b=\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)$$

and

$$y_2(t)=\frac{1}{a^2+b^2}+\frac{e^{at}}{b\sqrt{a^2+b^2}}\left(\sin\left[bt-\phi\right]\right)$$

$$a=-\frac{3}{2} \qquad b=\frac{\sqrt{3}}{2} \qquad \phi=\tan^{-1}\left(\frac{b}{a}\right)$$

4.29 (a)

$$\frac{d\,\Delta y}{dt}=B\,\Delta F_1-C\,\Delta F_2$$

$$\Delta y=y-\bar{y} \quad \Delta F_1=F_1-\bar{F} \quad \Delta F_2=F_2-\bar{F}$$

(b)

$$\frac{d\,\Delta y}{dt}=B\,\Delta y+C\,\Delta u$$

$$\Delta y=y-\bar{y} \quad \Delta u=u-\bar{u}$$

(c)

$$\frac{d\,\Delta y}{dt} = B\,\Delta y + C\,\Delta u$$
$$\Delta y = y - \bar{y} \quad \Delta u = u - \bar{u}$$

Note that D does not appear in the differential equation in deviation form.

4.30 **(a)** First write the equation in deviation variable form and then apply Laplace transforms to each term in the equation which yields

$$s^2\,Y(s) + 5s\,Y(s) + 6Y(s) = U(s)$$

Note that $y(0)$ can be assumed equal to zero for deviation variables. Rearranging

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)}$$

(b) First write the equation in deviation variable form and then apply Laplace transforms to each term in the equation which yields

$$s^2\,Y(s) + 3s\,Y(s) = U(s)$$

Note that $y(0)$ can be assumed equal to zero for deviation variables. Rearranging

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+3)}$$

(c) First write the equation in deviation variable form and then apply Laplace transforms to each term in the equation which yields

$$s^2\,Y(s) + 4s\,Y(s) - 5Y(s) = U(s)$$

Note that $y(0)$ can be assumed equal to zero for deviation variables. Rearranging

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+4s-5} = \frac{1}{(s+5)(s-1)}$$

(d) First write the equation in deviation variable form and then apply Laplace transforms to each term in the equation which yields

$$s^2\,Y(s) + 4s\,Y(s) + 4Y(s) = U(s)$$

Note that $y(0)$ can be assumed equal to zero for deviation variables. Rearranging

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 4s + 4} = \frac{1}{(s+2)^2}$$

(e) First write the equation in deviation variable form and then apply Laplace transforms to each term in the equation which yields

$$2s^2\,Y(s) + 11s\,Y(s) + 12\,Y(s) = 5U(s)$$

Note that $y(0)$ can be assumed equal to zero for deviation variables. Rearranging

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5}{2s^2 + 11s + 12} = \frac{5}{(s+4)(2s+3)}$$

(f) First write the equation in deviation variable form and then apply Laplace transforms to each term in the equation which yields

$$3s^2\,Y(s) + 7\,s\,Y(s) + 2\,Y(s) = U(s)$$

Note that $y(0)$ can be assumed equal to zero for deviation variables. Rearranging

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{3s^2 + 7\,s + 2} = \frac{1}{(s+2)(3s+1)}$$

4.31 (a) Since the only pole is real and negative, the dynamic response of this process is stable and will involve dynamics of the form exp(-0.5 *t*).

(b) Since the poles are both real and negative, the dynamic response of this process is stable and will involves dynamics a linear combination of terms of the form exp(- *t*) and exp(-4t).

(c) Since the poles are imaginary with negative real parts, the dynamic response of this process is stable and will involves a damped oscillatory behavior.

(d) Since the poles are imaginary with positive real parts, the dynamic response of this process is unstable and will involves an underdamped oscillatory behavior.

(e) Since the poles are $s=\pm 3i$, sustained oscillations result with a frequency of 3.

(f) Since the poles are imaginary with negative real parts, the dynamic response of this process is stable and will involves a damped oscillatory behavior.

$$G(s)=\frac{T_s(s)}{F_1(s)}=\frac{F_1(T_1-\overline{T})}{M\,s+F_1+F_2}\cdot\frac{1}{\tau_{Ts}s+1}$$

4.38 For Equation 3.10, there is one nonlinear term that require linearization: F_1C

Linearizing this term yields $F_1C=\overline{F}_1\overline{C}+F_1(C-\overline{C})+C(F_1-\overline{F}_1)$

Substituting this result into Equation 3.9 into deviation variable form considering only C and F_1 change yields

$$M\frac{d\,\Delta C}{dt}=C_1\Delta F_1-(\overline{F}_1+F_2)\Delta C-\overline{C}\Delta F_1$$

Taking the Laplace transform of each term

$$M\,sC(s)=C_1F_1(s)-(\overline{F}_1+F_2)C(s)-\overline{C}F_1(s)$$

Solving for $T(s)/F_1(s)$ yields $G(s)=\dfrac{C(s)}{F_1(s)}=\dfrac{F_1(C_1-\overline{C})}{M\,s+F_1+F_2}$

Using the result from Problem 4.34 $\dfrac{C_s(s)}{C(s)}=e^{-\theta_A s}$

Then, the desired transfer function is the product of the individual transfer functions

$$G(s)=\frac{C_s(s)}{F_1(s)}=\frac{F_1(C_1-\overline{C})e^{-\theta_A s}}{M\,s+F_1+F_2}$$

4.39 From the properties of block diagram algebra

$$U(s)G_1(s)G_2(s)+U(s)G_3(s)+U(s)=Y(s)$$

$$\text{Re}\,arranging\quad\frac{Y(s)}{U(s)}=G_1(s)G_2(s)+G_3(s)+1$$

4.40 From the properties of block diagram algebra

$$E(s)K_c+E(s)K_c\tau_D\frac{1}{\tau_I s}=Y(s)$$

$$\text{Re}\,arranging\quad\frac{Y(s)}{E(s)}=K_c\left[1+\frac{\tau_D}{\tau_I}\right]$$

4.41 From the properties of block diagram algebra

$$[U(s)-Y(s)]G(s)=Y(s)$$

$$\text{Re}\,arranging\ \ \frac{Y(s)}{U(s)}=\frac{G(s)}{G(s)+1}$$

4.42 From the properties of block diagram algebra

$$U(s)G_1(s)G_2(s)+U(s)G_3(s)+G_4U(s)=Y(s)$$

$$\text{Re}\,arranging\ \ \frac{Y(s)}{U(s)}=G_1(s)G_2(s)+G_3(s)+G_4(s)$$