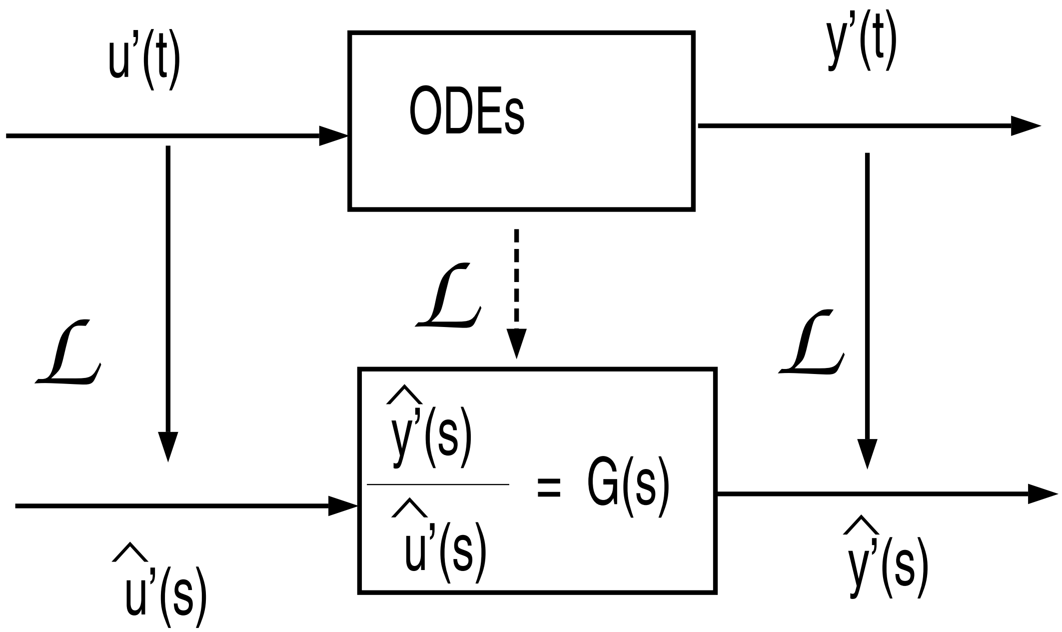
1 LAPLACE TRANSFER FUNCTION MODEL

In the "far" planet of Laplace, they didn't speak calculus.

Transfer Function Representation of I/O Relationship



$\hat{u}^{'}(s)$: Laplace transform of $u^{'}(t)$
$\hat{y}^{'}(s)$: Laplace transform of $y^{'}(t)$
$G(s)$: transfer function between $u^{'}(s)$ and $y^{'}(s)$

$G(s) \triangleq \frac{\hat{y}^{'}(s)}{\hat{u}^{'}(s)}$ is a compact, convenient way to represent the input / output relationship given by an ODE and can be easily obtained by *Laplace transforming the ODE.*

Quick Overview of Laplace Transform

Definition

Let $f(t)$ be a function of time (e.g., $u^{'}(t), y^{'}(t), \frac{dy^{'}}{dt}$, etc.). Then,

$$\hat{f}(s) = \mathcal{L}\{f(t)\} \triangleq \int_0^\infty f(t)e^{-st}dt$$

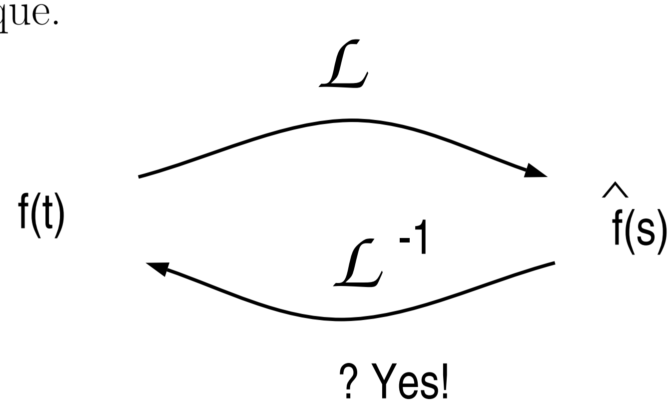
$\hat{f}(s)$ is called "Laplace transform of $f(t)$". Hence, Laplace transform "transforms" a function of $t \in \mathcal{R}^{0+}$ (i.e., t belonging to the set of nonnegative real numbers) into a function of $s \in \mathcal{C}$. Also, we may represent the "inverse Laplace transform" as follows:

$$\mathcal{L}^{-1}\{\hat{f}(s)\} = f(t)$$

- In the above definition, $\hat{f}(s)$ contains no information about $f(t)$ for $t < 0$. This means $\mathcal{L}^{-1}\{\hat{f}(s)\}$ is not well-defined for $t < 0$. This doesn't matter as we are interested in the future behavior and we can always make the starting time to be $t = 0$.
- Laplace transform is a linear operation:

$$\begin{aligned} \mathcal{L}\{af_1(t) + bf_2(t)\} &= a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_2(t)\} \\ &= a\hat{f}_1(s) + b\hat{f}_2(s) \end{aligned}$$

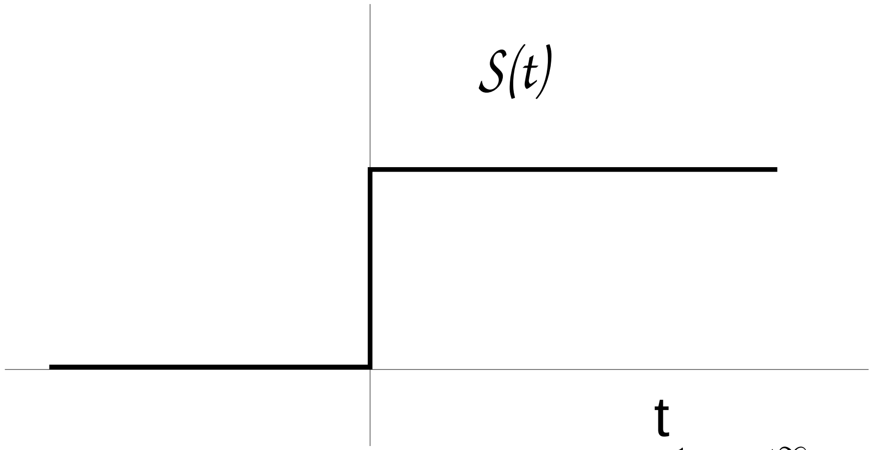
- Laplace transform is one-to-one mapping. Inverse always exists and is unique.



Some Examples – Learning the Vocabulary

1. Constant functon (step function)

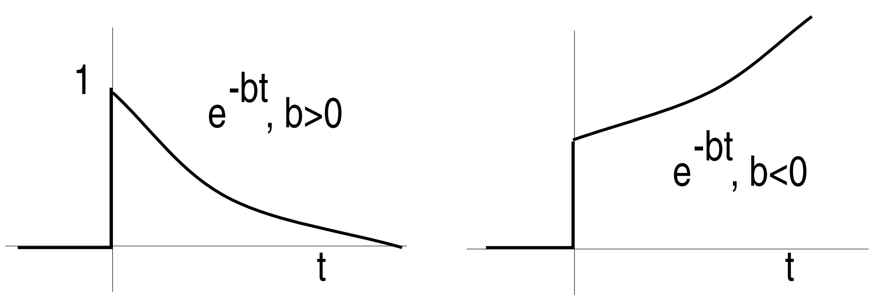
$$f(t) = \mathcal{S}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\begin{aligned} \mathcal{L}\{\mathcal{S}(t)\} &= \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^\infty \\ &= \frac{1}{s} \end{aligned}$$

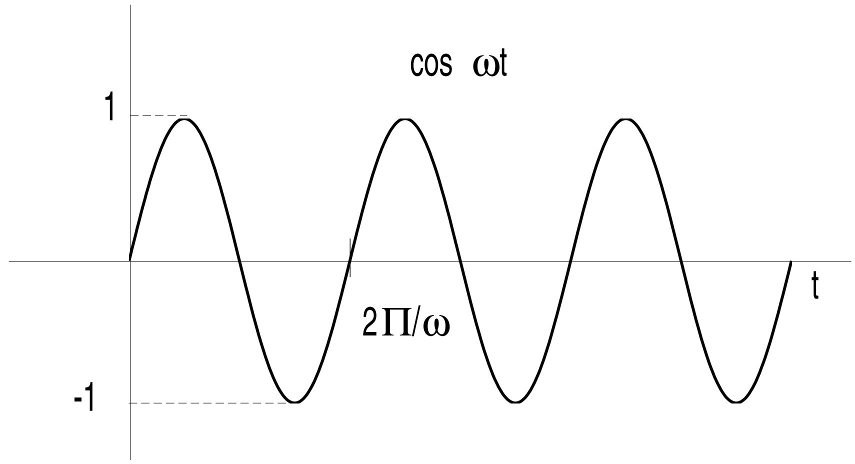
2. Exponential function

$$f(t) = e^{-bt}$$



$$\begin{aligned} \mathcal{L}\{e^{-bt}\} &= \int_0^\infty e^{-bt} e^{-st} dt = \int_0^\infty e^{-(b+s)t} dt \\ &= \frac{-1}{s+b} e^{-(b+s)t}\Big|_0^\infty = \frac{1}{s+b} \end{aligned}$$

3. Trignometric functions



$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Note that

$$\begin{aligned} e^{j\omega t} &\triangleq \cos\omega t + j\sin\omega t \\ e^{-j\omega t} &\triangleq \cos\omega t - j\sin\omega t \end{aligned}$$

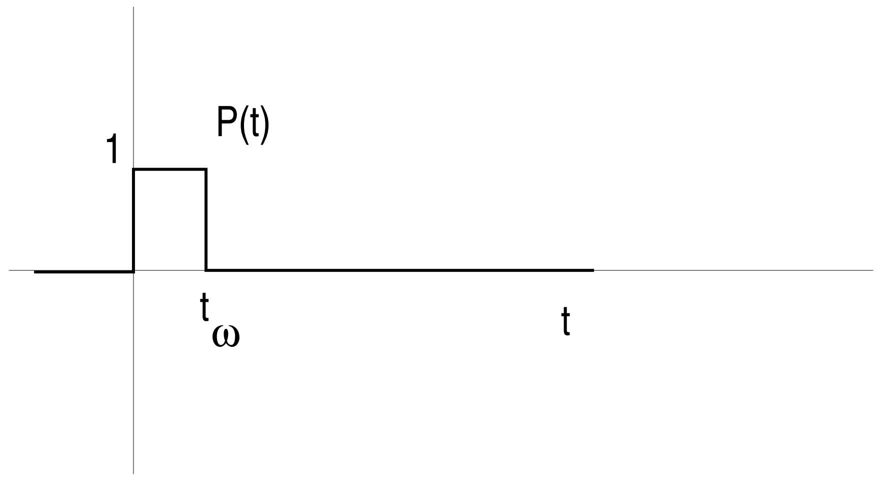
$$\begin{aligned} \mathcal{L}\{\cos\omega t\} &= \mathcal{L}\left\{\frac{e^{j\omega t}}{2}\right\} + \mathcal{L}\left\{\frac{e^{-j\omega t}}{2}\right\} \\ &= \frac{1}{2}\left(\frac{1}{s-j\omega}\right) + \frac{1}{2}\left(\frac{1}{s+j\omega}\right) \\ &= \frac{1}{2}\left(\frac{2s}{s^2+\omega^2}\right) \\ &= \frac{s}{s^2+\omega^2} \end{aligned}$$

Similarly,

$$\mathcal{L}\{\sin\omega t\} = ????$$

4. Rectangular Pulse Functions

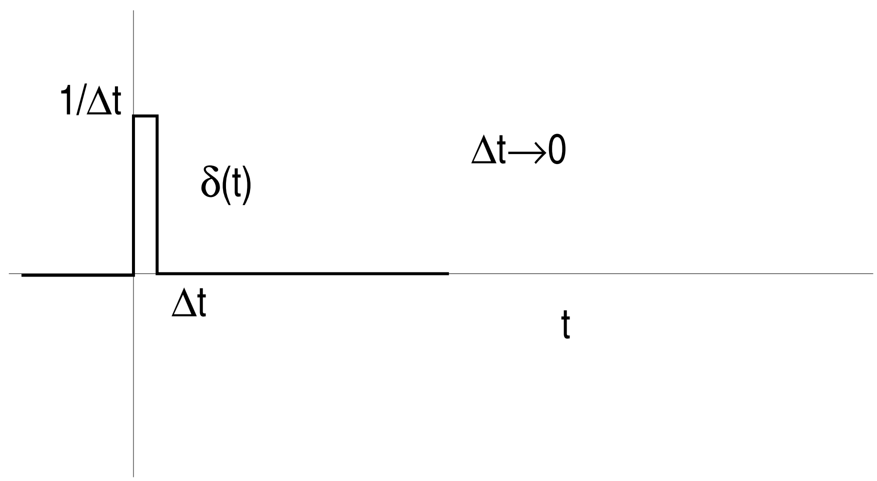
$$f(t) = \mathcal{P}(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \leq t < t_w \\ 0 & t \geq t_w \end{cases}$$



$$\begin{aligned} \mathcal{L}\{\mathcal{P}(t)\} &= \int_0^\infty \mathcal{P}(t) e^{-st} dt \\ &= \int_0^{t_w} e^{-st} dt \\ &= -\frac{1}{s} e^{-st} \Big|_0^{t_w} \\ &= \frac{1}{s}(1 - e^{-t_w s}) \end{aligned}$$

5. Impulse Function (Idealization of pulse of short duration)

$$f(t) = \delta(t) = \lim_{t_w \to 0} \begin{cases} 0 & t < 0 \\ \frac{1}{t_w} & 0 \le t < t_w \\ 0 & t \ge t_w \end{cases}$$

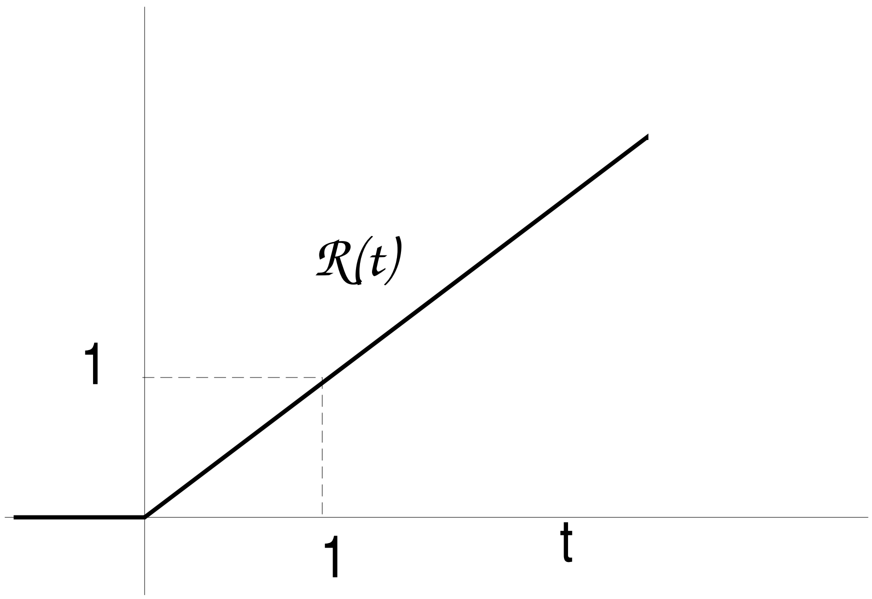


$$\begin{aligned} \mathcal{L}\{\delta(t)\} &= \lim_{t_w \to 0} \frac{1}{t_w s}\left(1 - e^{-t_w s}\right) \\ &= \lim_{t_w \to 0} \left(\frac{s e^{-t_w s}}{s}\right) \text{ [L'Hopital's Rule]} \\ &= 1 \end{aligned}$$

Impulse of area a: $\mathcal{L}\{a\delta(t)\} = a$

6. Ramp function

$$f(t) = \mathcal{R}(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$



$$\begin{aligned}
\mathcal{L}\{\mathcal{R}(t)\} &= \int_0^\infty te^{-st}dt \\
&= \frac{t}{(-s)}e^{-st}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt \qquad \text{[intg. by parts]} \\
&= 0 - 0 + \frac{1}{s}\int_0^\infty e^{-st}dt \\
&= \frac{1}{s}\left(\frac{1}{s}\right) \\
&= \frac{1}{s^2}
\end{aligned}$$

Note: Integration by Parts:

$$\int_0^\infty h' \cdot g dt = h \cdot g|_0^\infty - \int_0^\infty h \cdot g' dt$$

In the above, we chose $h = \frac{e^{-st}}{-s}$ and $g = t$.

Note: See Table C.1 for more transforms.

Some Important Properties – Learning the Grammar

1. Differentiation

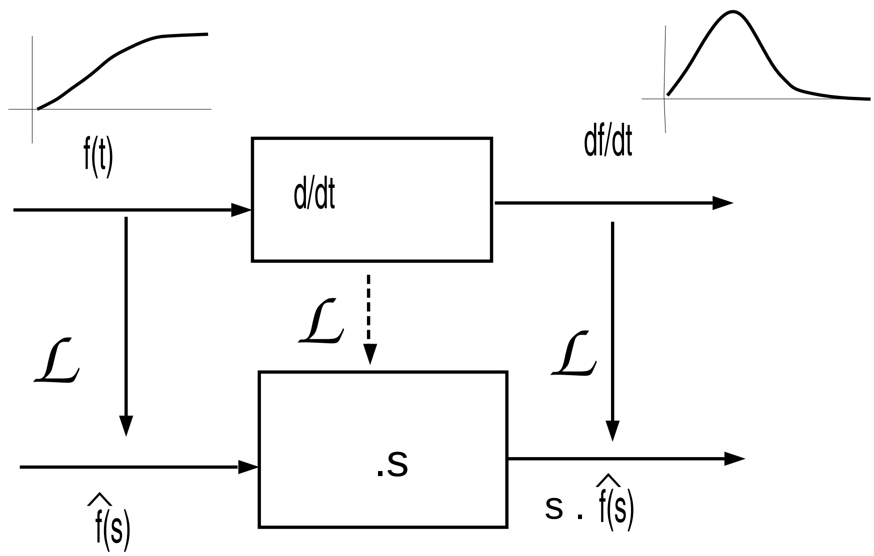
$$
\begin{aligned}
\mathcal{L}\left\{\frac{df}{dt}\right\} &= \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \\
&= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st}dt \text{ [intg. by parts]} \\
&= 0 - f(t)|_{t=0} + s\int_0^\infty f(t)e^{-st}dt \\
&= s \cdot \hat{f}(s) - f(t)|_{t=0}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} &= \mathcal{L}\left\{\frac{d}{dt}\left(\frac{df}{dt}\right)\right\} \\
&= s \cdot \mathcal{L}\left\{\frac{df}{dt}\right\} - \frac{df}{dt}|_{t=0} \\
&= s\left(s \cdot \hat{f}(s) - f(t)|_{t=0}\right) - \frac{df}{dt}|_{t=0} \\
&= s^2 \cdot \hat{f}(s) - s \cdot f(t)|_{t=0} - \frac{df}{dt}|_{t=0}
\end{aligned}
$$

Continue this to get Laplace transform of any order derivative. For instance,

$$
\mathcal{L}\left\{\frac{d^3 f}{dt^3}\right\} = s^3\hat{f}(s) - s^2 f(t)|_{t=0} - s\frac{df}{dt}|_{t=0} - \frac{d^2 f}{dt^2}|_{t=0}
$$

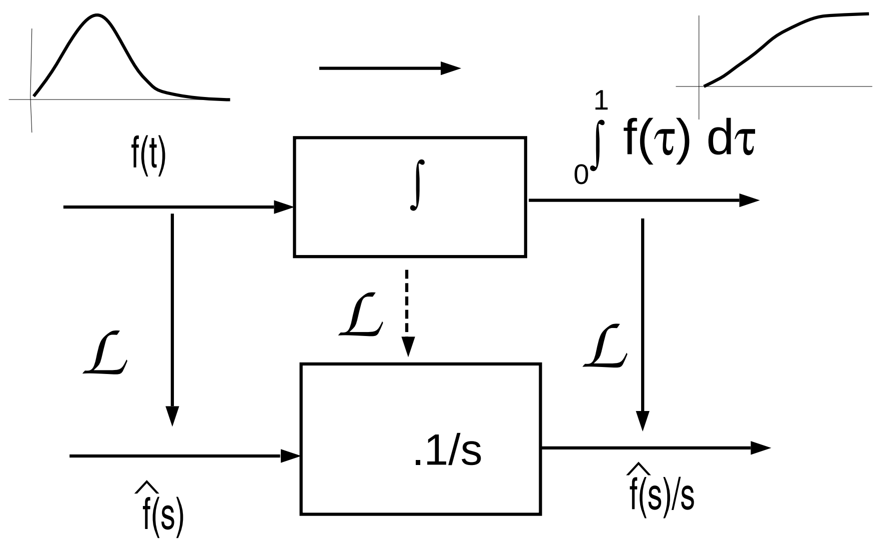
Differentiation in time domain is equivalent to multiplication by s in Laplace domain!



2. Integration

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} &= \int_0^\infty \left[\int_0^t f(\tau)d\tau\right] e^{-st}dt \\
&= \frac{e^{-st}}{(-s)}\int_0^t f(\tau)d\tau\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{(-s)} f(t)dt \\
&= 0 + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt \\
&= \frac{\hat{f}(s)}{s}
\end{aligned}$$

Integration in time domain is equivalent to dividing by s in Laplace domain!



NOTE: Leibniz Rule

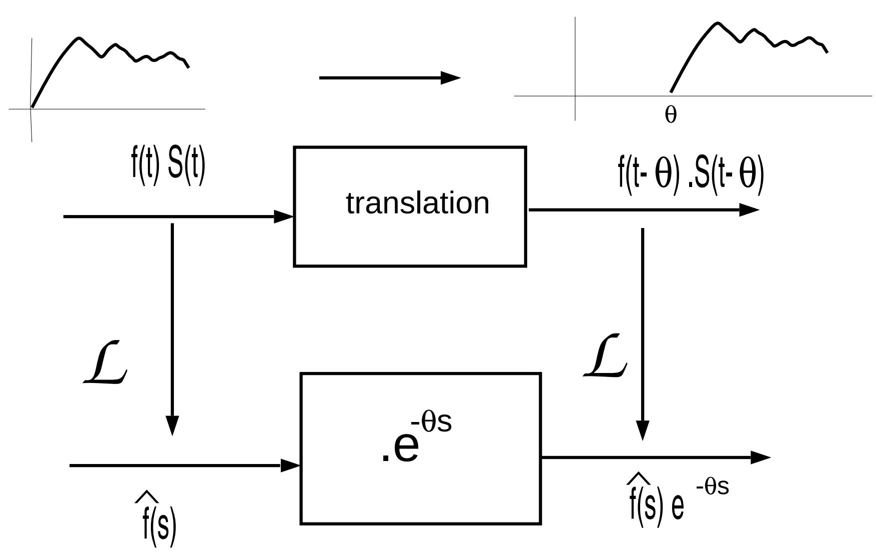
$$\frac{d}{dt}\left(\int_{u(t)}^{v(t)} f(\tau)d\tau\right) = f(v(t))\frac{dv}{dt} - f(u(t))\frac{du}{dt}$$

This should be apparent from Chain Rule.

3. Translation (delay)

$$
\begin{aligned}
\mathcal{L}\{f(t-\theta)\mathcal{S}(t-\theta)\} &= \int_0^\infty f(t-\theta)\mathcal{S}(t-\theta)e^{-st}dt \\
&= \int_\theta^\infty f(t-\theta)e^{-st}dt \\
&= \int_0^\infty f(t^{'})e^{-s(t^{'}+\theta)}dt^{'} \text{ [where } t^{'} = t-\theta] \\
&= \left(\int_0^\infty f(t^{'})e^{-st^{'}}dt^{'}\right) e^{-s\theta} \\
&= \hat{f}(s)\cdot e^{-s\theta}
\end{aligned}
$$

Translation by θ (of function $f(t)$ whose value is zero for $t < 0$) is equivalent to multiplication by $e^{-\theta s}$ in Laplace domain!



Finding Initial and Final Values from Laplace Transform

Problem: Given $\hat{f}(s)$, find $f(t)$ at $t = 0$ and $t = \infty$.

- Final Value Theorem:

$$f(\infty) = \lim_{s \to 0} \left(s \cdot \hat{f}(s) \right)$$

(Proof)

$$\begin{aligned}
\int_0^\infty \frac{df}{dt} e^{-st} dt &= s \cdot \hat{f}(s) - f(0) \\
&\Downarrow \\
\lim_{s \to 0} \left(\int_0^\infty \frac{df}{dt} e^{-st} dt \right) &= \lim_{s \to 0} \left(s \cdot \hat{f}(s) - f(0) \right) \\
&\Downarrow \\
f(\infty) - f(0) &= \lim_{s \to 0} \left(s \cdot \hat{f}(s) - f(0) \right) \\
&\Downarrow \\
f(\infty) &= \lim_{s \to 0} s \cdot \hat{f}(s)
\end{aligned}$$

Note that the theorem does NOT apply when $y(t)$ does not converge as $t \to \infty$ (since the Laplace transform is not defined for $s = 0$ in this case). The theorem cannot be used to distinguish between converging and diverging signals.

- Initial Value Theorem:

$$f(0) = \lim_{s \to \infty} \left(s \cdot \hat{f}(s) \right)$$

(Proof) Similar to the previous proof. Try and see if you can do it.

Examples:

1.

$$\hat{f}(s) = \frac{2}{s+5}$$

$$f(\infty) = \lim_{s\to 0} \frac{2s}{s+5} = 0$$

$$f(0) = \lim_{s\to \infty} \frac{2s}{s+5} = 2$$

2.

$$\hat{f}(s) = \frac{5}{s^2+3s}$$

$$f(\infty) = \lim_{s\to 0} \frac{5s}{s^2+3s} = \frac{5}{3}$$

$$f(0) = \lim_{s\to \infty} \frac{5s}{s^2+3s} = 0$$

3.

$$\hat{f}(s) = \frac{2}{s-5}$$

Note in this case $f(t) = 2e^{5t}$.

$$f(\infty) = \infty, \text{ but } \lim_{s\to 0} \frac{2s}{s-5} = 0$$

$$f(0) = \lim_{s\to \infty} \frac{2s}{s+5} = 2$$

Inverse Laplace Transform (Back Translation)

Problem: Given $\hat{f}(s)$, find $f(t)$ such that

$$\mathcal{L}\{f(t)\} = \hat{f}(s)$$

We denote this operation as

$$f(t) = \mathcal{L}^{-1}\{\hat{f}(s)\}$$

The simplest method to find $f(t)$ (besides looking up the table) is to take partial fraction expansion of $\hat{f}(s)$ and compute the inverse of each factor:

$$\begin{aligned}\hat{f}(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s+a_1)\cdots(s+a_n)} \\ &= \frac{\alpha_1}{s+a_1} + \cdots + \frac{\alpha_n}{s+a_n}\end{aligned}$$

To compute the coefficient α_i, multiply both sides by $(s+a_i)$ and evaluate at $s=-a_i$:

$$\alpha_i = \frac{N(s)}{(s+a_1)\cdots(s+a_n)}(s+a_i)\bigg|_{s=-a_i}$$

Once the coefficients are found, the inverse Laplace transform is easy:

$$f(t) = \alpha_1 e^{-a_1 t} + \cdots + \alpha_n e^{-a_n t}$$

Some complications are

- when $D(s)$ contains repeated factors (i.e., same factor several times)
- when $D(s)$ contains complex conjugate factors

Examples: To be given.

Solving Linear ODEs via Laplace Transform

Key property to use is

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\hat{y}(s) - y(0)$$

This property enables conversion of a differential equation into an algebraic equation by taking Laplace transform:

Examples:

1.
$$5\frac{dy}{dt} + 4y = 2; \quad y(0) = 1$$

Take Laplace transform of both sides:

$$\mathcal{L}\left\{5\frac{dy}{dt} + 4y\right\} = \mathcal{L}\left\{2\right\}$$

$$5\left(s\hat{y}(s) - y(0)\right) + 4\hat{y}(s) = \frac{2}{s}$$

$$(5s+4)\hat{y}(s) - 5y(0) = \frac{2}{s}$$

$$(5s+4)\hat{y}(s) = \frac{2}{s} + 5$$

$$\begin{aligned}\hat{y}(s) &= \frac{(5s+2)}{s(5s+4)} \\ &= \frac{1/2}{s} + \frac{5/2}{5s+4}\end{aligned}$$

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left\{\frac{1/2}{s} + \frac{5/2}{5s+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/2}{s} + \frac{1/2}{s+4/5}\right\} \\ &= \frac{1}{2} + \frac{1}{2}e^{-0.8t}\end{aligned}$$

Notice we solved the differential equation without any integration!

2.

$$\frac{d^3y}{dt^3} + 6\frac{d^2}{dt^2} + 11\frac{dy}{dt} + 6y = 3t; \quad \frac{dy}{dt}(0) = \frac{dy}{dt}(0) = 0, y(0) = 1$$

Try solving this using Laplace transform.

Point to Think About: Laplace transform technique only works with linear ODEs. It cannot be used to solve ODEs with nonlinear terms. Why?

Deriving Transfer Function from I/O ODEs

ODE models used for control contains input u and output y. For instance, first-order model is of the form

$$\tau \frac{dy^{'}}{dt} + y^{'} = ku^{'}$$

Of course, once $u^{'}(t)$ (and the initial condition) is specified, one can use the Laplce transform technique to solve for $y^{'}(t)$. However, it is often convenient to leave $u^{'}$ in the expression. Then, Laplace transform of the ODE model (with zero initial condition) gives an algebraic equation that contains both $\hat{y}^{'}(s)$ and $\hat{u}^{'}(s)$. Through simple algebra, one can put this equation into the form

$$\frac{\hat{y}^{'}(s)}{\hat{u}^{'}(s)} = G(s)$$

where $G(s)$ is a rational function of s. $G(s)$ is called "transfer function" between $\hat{u}^{'}(s)$ and $\hat{y}^{'}(s)$ (function that "transfers" input transform to output transform).

Examples:

1.

$$\tau \frac{dy^{'}}{dt} + y^{'} = ku^{'}$$

$$\tau \left(s\hat{y}^{'}(s) - y^{'}(0) \right) + \hat{y}^{'} = k\hat{u}^{'}(s)$$

With $y^{'}(0) = 0$, we get

$$(\tau s + 1)\hat{y}^{'}(s) = k\hat{u}^{'}(s)$$

$$\frac{\hat{y}^{'}(s)}{\hat{u}^{'}(s)} = \frac{k}{\tau s + 1}$$

Notice $\hat{y}'(s)$ is obtained by multiplying $\hat{u}'(s)$ with $G(s)$.

2.

$$\tau^2 \frac{d^2 y'}{dt^2} + 2\tau\zeta \frac{dy'}{dt} + y' = ku'$$

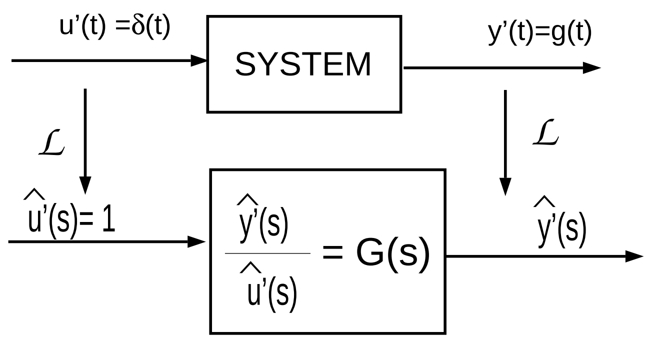
$$\tau^2 \left(s^2 \hat{y}'(s) - s y'(0) - \frac{dy'}{dt}(0) \right) + 2\tau\zeta \left(\hat{y}'(s) - y'(0) \right) + \hat{y}'(s) = k\hat{u}'(s)$$

With $y'(0) = \frac{dy'}{dt}(0) = 0$, we get

$$(\tau^2 s^2 + 2\zeta\tau + 1)\hat{y}'(s) = k\hat{u}'(s)$$

$$\frac{\hat{y}'(s)}{\hat{u}'(s)} = \frac{k}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Point to Think About: Transfer function is the Laplace transform of the impulse response of the system. Why?



Compare with the impulse response model presented earlier.

$$y'(t) = \int_0^t g(t-\tau)u'(\tau)d\tau$$

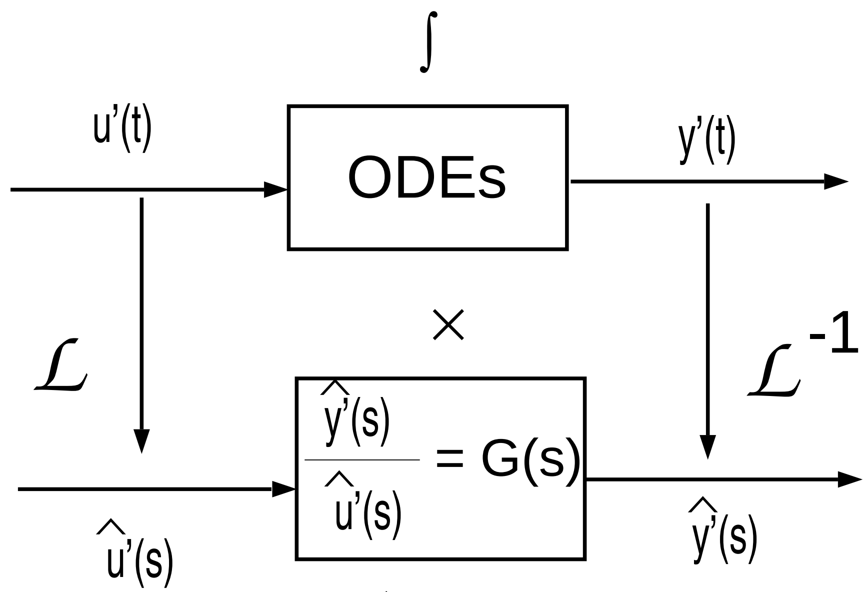
Convolution integral became simple multiplication.

$$\hat{y}'(t) = G(s) \cdot \hat{u}'(s) = \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{u'(t)\}$$

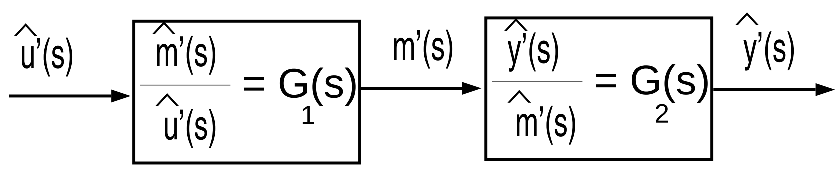
Why Transfer Function Representation?

The transfer function representation is preferred over the ODE representation in mathematical analysis since

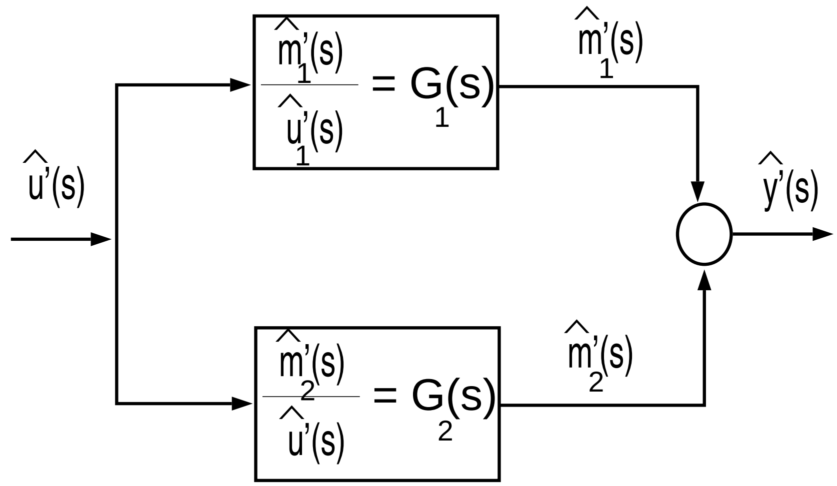
- for given $u^{'}(t)$, the ODE must be solved (requiring integration) to find $y^{'}(t)$, while we can compute $y^{'}(s)$ easily via $y^{'}(s) = G(s)u^{'}(s)$.



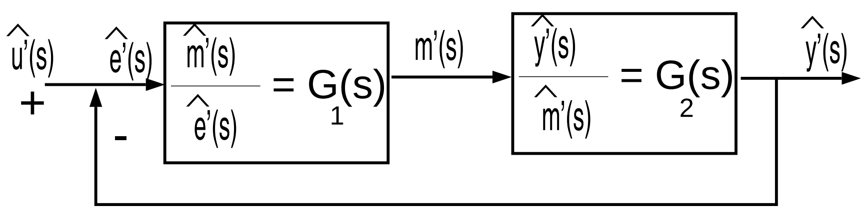
- it is simpler to compute I/O relations for connected systems from I/O relations of individual blocks.



$$\frac{\hat{y}^{'}(s)}{\hat{u}^{'}(s)} = G_1(s) \cdot G_2(s)$$



$$\frac{\hat{y}'(s)}{\hat{u}'(s)} = G_1(s) + \cdot G_2(s)$$



$$\frac{\hat{y}'(s)}{\hat{u}'(s)} = \frac{G_1(s) \cdot G_2(s)}{1 + G_1(s) \cdot G_2(s)}$$

- It is easier to analyze.

$$G(s) = \frac{\hat{y}'(s)}{\hat{u}'(s)} = \frac{N(s)}{D(s)}$$

  Roots of $D(s) = 0$ are called "poles" and roots of $N(s) = 0$ are called "zeros". One can easily find out the qualitative aspects of a given I/O relationship (e.g., fast vs. slow, overdamped vs. underdamped, inverse response vs. monotonic response) by examining the location of "poles" and "zeros" of the transfer function in the complex plane.

Some Examples on Using Transfer Function for I/O Analysis