

Lecture 7  
Closed-Loop Analysis  
Part I: Laplace Domain Analysis

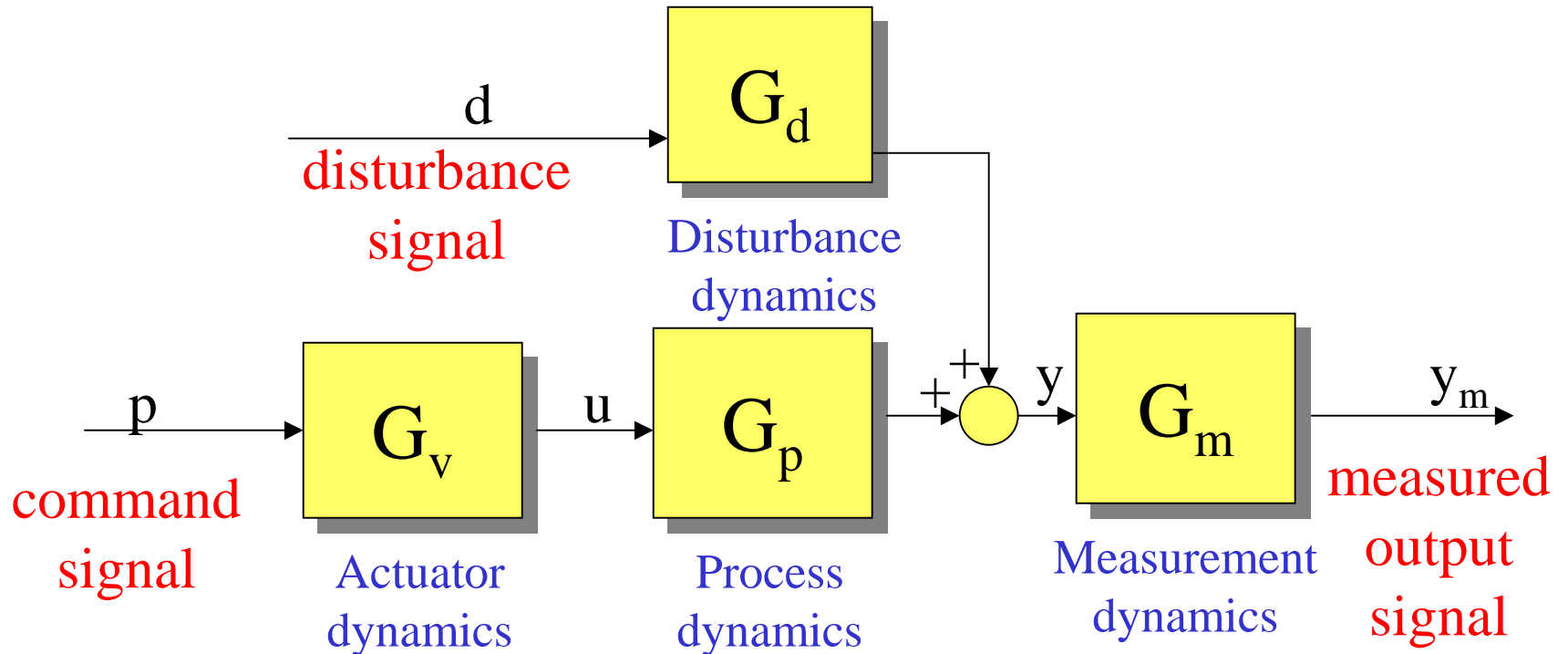
CHE4400

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# Overview



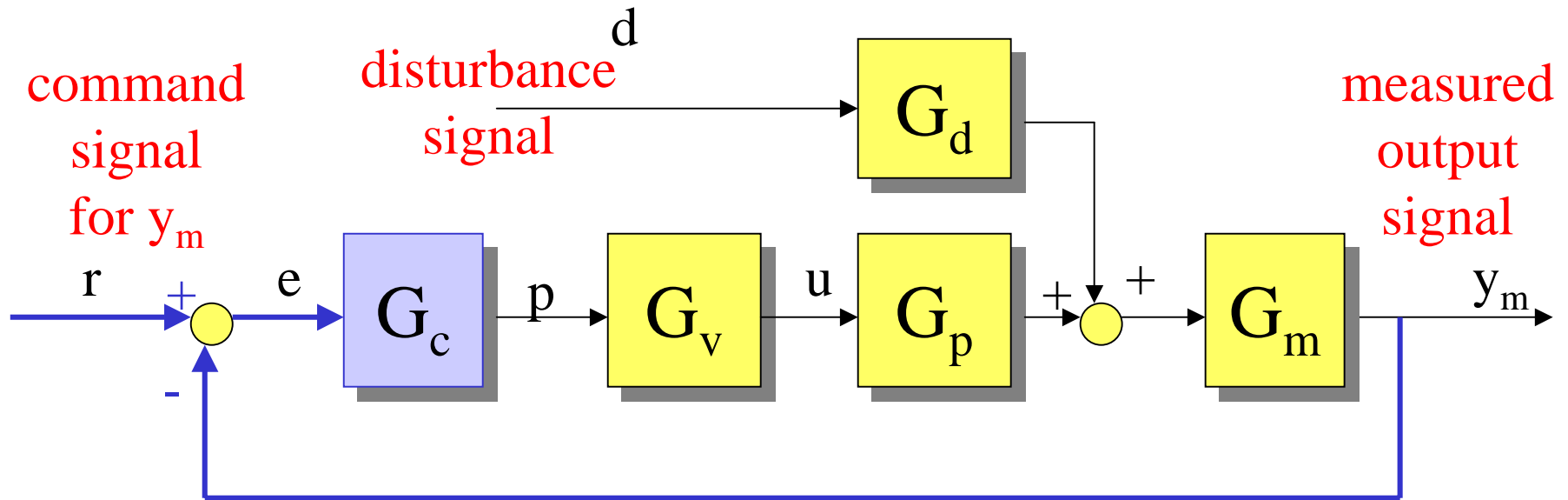
$$\frac{\hat{y}_m(s)}{\hat{p}(s)} = G_m G_p G_v$$

- The gain may not be 1
- The dynamics may be too slow, too oscillatory or unstable.

$$\frac{\hat{y}_m(s)}{\hat{d}(s)} = G_m G_d$$

- The gain may not be 0
- The dynamics may be too slow.....

# Overview



$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{G_m G_p G_v G_c}{1 + G_m G_p G_v G_c} = \frac{GG_c}{1 + GG_c}$$

$G$

$$\frac{\hat{y}_m(s)}{\hat{d}(s)} = \frac{G_m G_d}{1 + G_m G_p G_v G_c} = \frac{G_{md}}{1 + GG_c}$$

$G_{md}$

- Different from open-loop!
- Depends on  $G_c$

# Analysis and Design Problems

- Analysis: Given particular  $G$ ,  $G_{md}$  and  $G_c$ 
  - Are the closed-loop dynamics stable? ← *Locations of poles and zeros of the closed-loop transfer functions*
  - Speed of response? Damping? ← *Locations of poles and zeros of the closed-loop transfer functions*
  - Gains for  $y_m/r$  and  $y_m/d$
- Design: Given particular  $G$  and  $G_{md}$ , choose (“design”)  $G_c$  so that
  - the closed-loop dynamics are stable.
  - $y_m/r$  has a gain of  $\sim 1$  and  $y_m/d$  has a gain of  $\sim 0$ .
  - the dynamics are sufficiently fast (but not too fast) and smooth (without excessive oscillation).

# Model Used for Analysis and Design

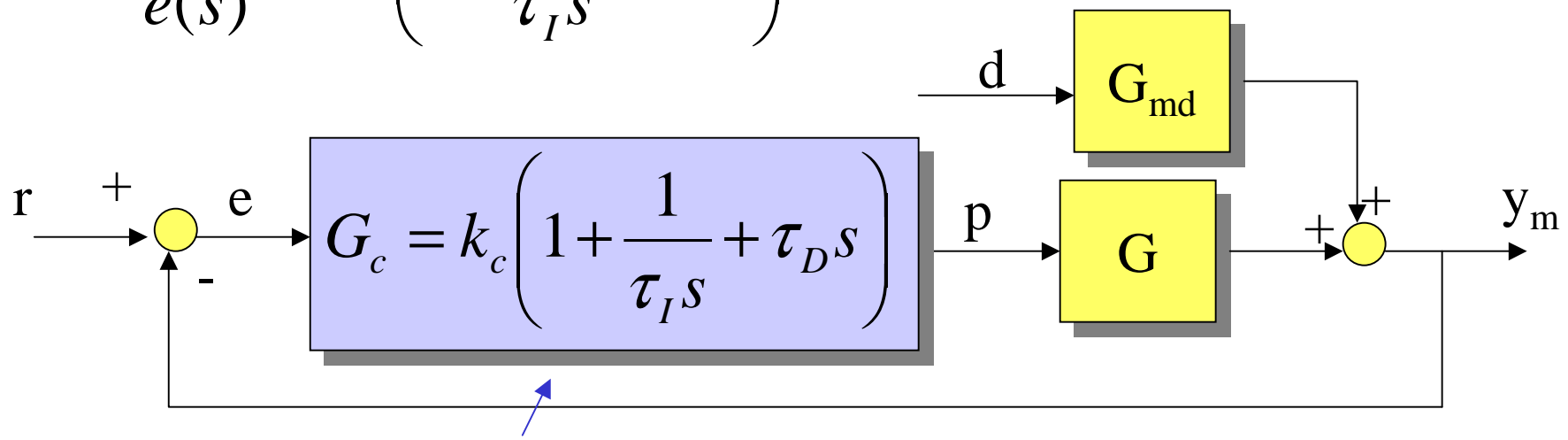
- Case I (Less Frequent)
  - From a fundamental model, perform linearization and Laplace transform of the linearized ODEs to find  $G_p(s)$  and  $G_d(s)$
  - Find actuator and measurement dynamics  $G_v$  and  $G_m$  to obtain  $G = G_m G_p G_v$  and  $G_{md} = G_m G_d$
- Case II (More Frequent)
  - The composite model  $G$  is fitted to data of  $y_m$  obtained by perturbing  $p$  (e.g., by making a step change).
  - $G_{md}$  is often assumed to be same as  $G$  (as in IAE/ISE/ITAE tuning rules).

# PID Controller

$$p(t) = \bar{p} + k_c \left( e(t) + \frac{1}{\tau_I} \int_0^t e(t') dt' + \tau_D \frac{de}{dt} \right) \Rightarrow$$

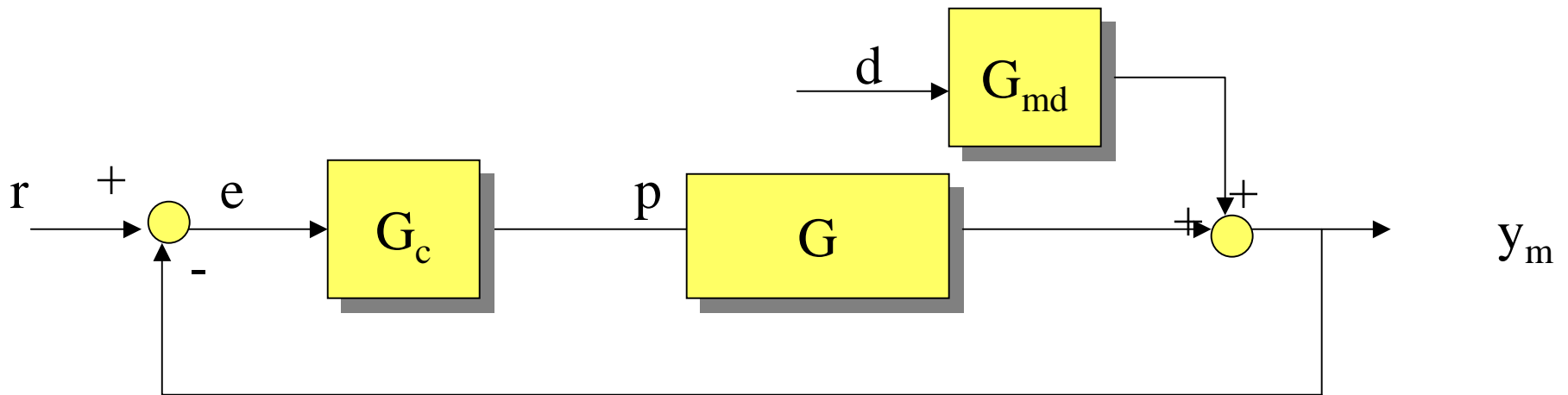
$$\hat{p}'(s) = k_c \left( \hat{e}(s) + \frac{1}{\tau_I s} \hat{e}(s) + \tau_D s \hat{e}(s) \right) \Rightarrow$$

$$\frac{\hat{p}'(s)}{\hat{e}(s)} = k_c \left( 1 + \frac{1}{\tau_I s} + \tau_D s \right)$$



However,  $G_c$  can be chosen as any transfer function in general.

# Calculation of Closed-Loop Functions



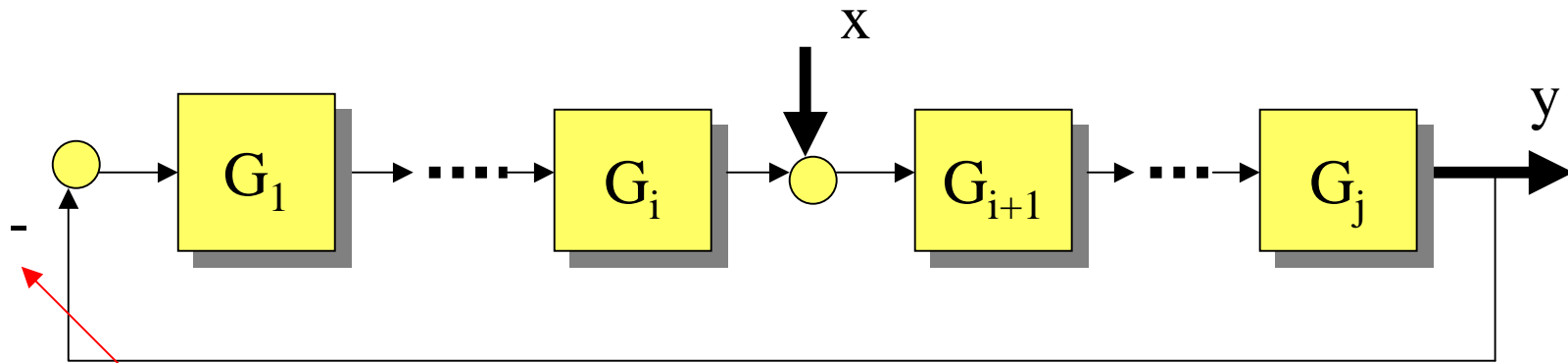
$$\hat{y}_m = G_{md} \hat{d} + G \hat{p} \text{ and } \hat{p} = G_c (\hat{r} - \hat{y}_m) \Rightarrow$$

$$\hat{y}_m = G_{md} \hat{d} + GG_c (\hat{r} - \hat{y}_m) \Rightarrow (1 + GG_c) \hat{y}_m = G_{md} \hat{d} + GG_c \hat{r}$$

$$\frac{\hat{y}_m(s)}{\hat{d}(s)} = \frac{G_{md}}{(1 + GG_c)} \text{ and } \frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{GG_c}{(1 + GG_c)}$$

See the convenience the Laplace transform affords us?

# Calculation of Closed-Loop Functions - General



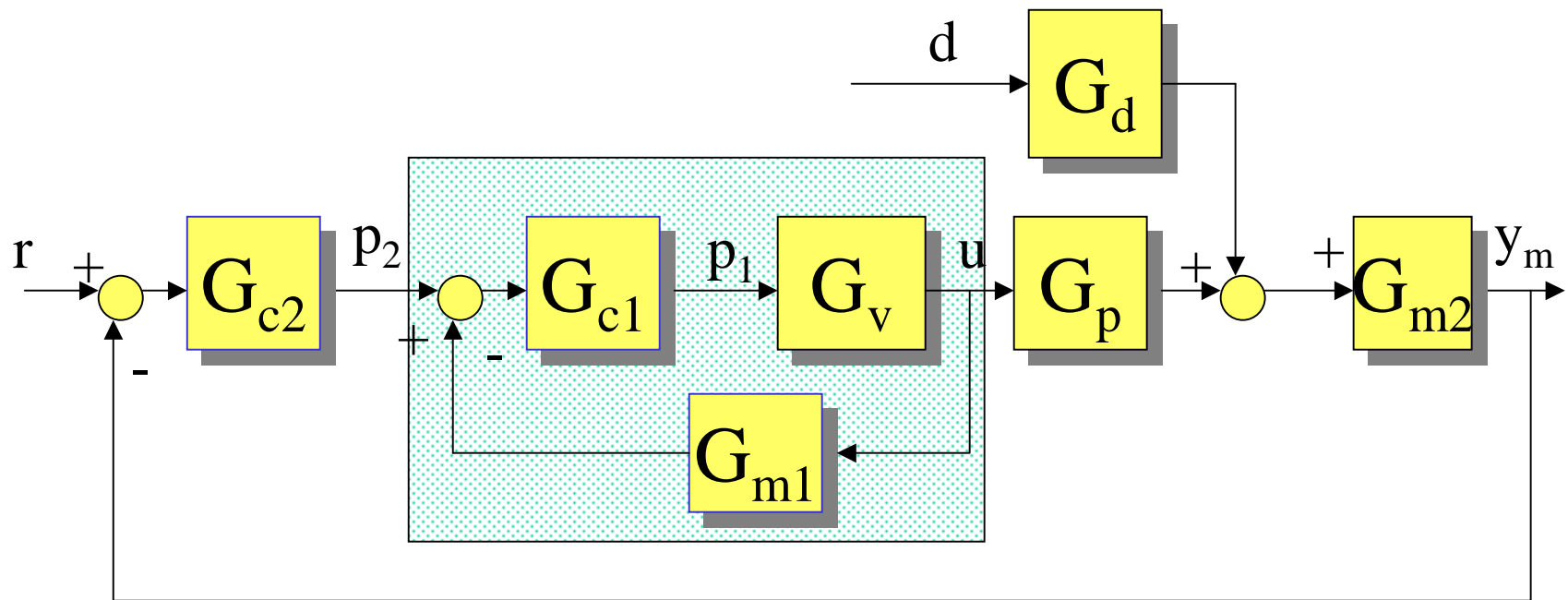
Assume “negative feedback”

open-loop path from  $x$  to  $y$

$$\frac{\hat{y}(s)}{\hat{x}(s)} = \frac{G_{i+1}G_{i+2} \cdots G_j}{1 + G_1G_2 \cdots G_j} = \frac{G_{OL}}{1 \oplus G_{CL}}$$

Product of all blocks inside the loop

# Handling of Cascaded Loops

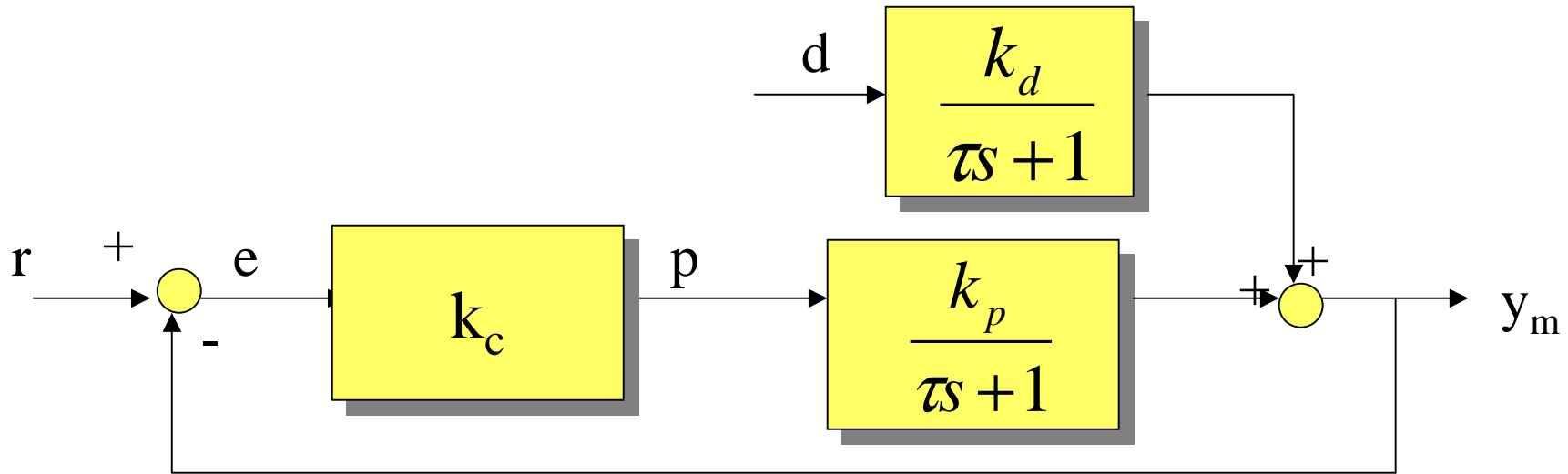


$$\frac{\hat{u}(s)}{\hat{p}_2(s)} = \frac{G_{c1}G_v}{1 + G_{c1}G_vG_{m1}} = G_{vnew}$$

New actuator dynamics  
modified by the inner loop!

$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{G_{m2}G_pG_{vnew}G_{c2}}{1 + G_{m2}G_pG_{vnew}G_{c2}}; \quad \frac{y_m(s)}{d(s)} = \frac{G_{m2}G_d}{1 + G_{m2}G_pG_{vnew}G_{c2}}$$

# Analysis of P-only Control

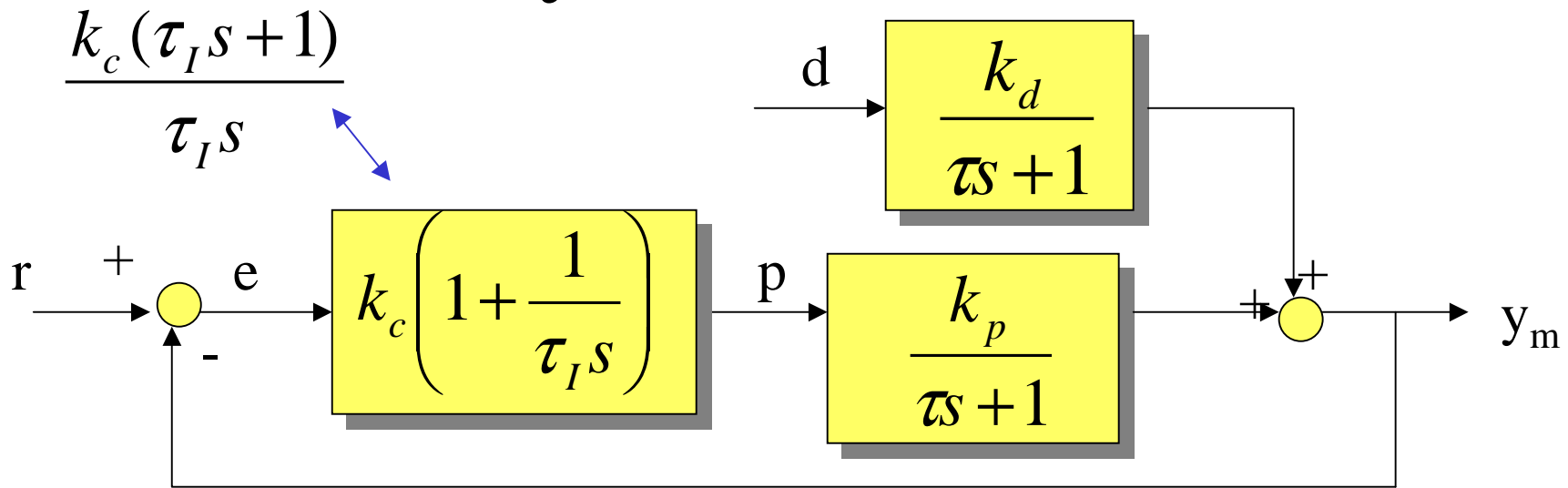


$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{\frac{k_c k_p}{\tau s + 1}}{1 + \frac{k_c k_p}{\tau s + 1}} = \frac{k_c k_p}{\tau s + 1 + k_c k_p} = \frac{\frac{k_c k_p}{1 + k_c k_p}}{\frac{\tau}{1 + k_c k_p} s + 1}$$

$$\frac{\hat{y}_m(s)}{\hat{d}(s)} = \frac{\frac{k_d}{\tau s + 1}}{1 + \frac{k_c k_p}{\tau s + 1}} = \frac{k_d}{\tau s + 1 + k_c k_p} = \frac{\frac{k_d}{1 + k_c k_p}}{\frac{\tau}{1 + k_c k_p} s + 1}$$

- Gain is not 1 unless  $k_c = \infty$
- Time constant decreases with increasing  $k_c$
- Gain is not 0 unless  $k_c = \infty$

# Analysis of PI Control



$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{\frac{k_c k_p (\tau_I s + 1)}{\tau_I s (\tau s + 1)}}{1 + \frac{k_c k_p (\tau_I s + 1)}{\tau_I s (\tau s + 1)}} = \frac{k_c k_p (\tau_I s + 1)}{\tau_I s (\tau s + 1) + k_c k_p (\tau_I s + 1)}$$

$$= \frac{(\tau_I s + 1)}{\frac{\tau_I \tau}{k_c k_p} s^2 + \frac{1 + k_c k_p}{k_c k_p} \tau_I s + 1}$$

- Gain = 1 always! No offset.
- 2<sup>nd</sup> order dynamics
- Underdamped dynamics for very small  $\tau_I$

# Characteristic Equation

$$1 + GG_c = 0$$

- Roots of the above equation are the poles of the closed-loop functions (important information for analyzing closed-loop dynamics)
- For stability, make sure all the roots are in the Left-Half-Plane (negative real parts)
  - Can be checked by Routh's test (14.6.3 of the textbook)
  - Or by Direct Substitution (14.6.4)

# Example: Routh's Test

Main Idea: Form a Routh array to see if any roots are in the RHP.

$$1 + \frac{6K_c}{(2s+1)(4s+1)(6s+1)} = 0 \Rightarrow$$

$$48s^3 + 44s^2 + 12s + (1 + 6K_c) = 0$$

$$\frac{44 \times 12 - 48(1 + 6K_c)}{44}$$

48	12	
44	$1 + 6K_c$	
$\frac{120}{11} - \frac{72}{11}K_c$		0
$1 + 6K_c$		

$$\frac{\left(\frac{120}{11} - \frac{72}{11}K_c\right) \times (1 + 6K_c) - 44 \times 0}{\left(\frac{120}{11} - \frac{72}{11}K_c\right)}$$

Must be all positive  
for closed-loop  
stability!

$$\frac{120}{11} - \frac{72}{11}K_c > 0 \Rightarrow K_c < \frac{5}{3}$$

$$1 + 6K_c > 0 \Rightarrow K_c > -\frac{1}{6}$$

# Example: Direct Substitution

Main Idea: At the limits of instability, the closed-loop poles will be on the imaginary axis (between LHP and RHP)

$$48s^3 + 44s^2 + 12s + (1 + 6K_c) = 0 \xrightarrow{s=j\omega}$$
$$-48j\omega^3 - 44\omega^2 + 12j\omega + (1 + 6K_c) = 0 \rightarrow$$
$$(-48\omega^3 + 12\omega)j + [-44\omega^2 + (1 + 6K_c)] = 0 \rightarrow$$
$$\begin{aligned} -48\omega^3 + 12\omega &= 0 & \omega &= 0, K_c = -1/6 \\ -44\omega^2 + (1 + 6K_c) &= 0 & \omega &= \pm 1/2, K_c = 5/3 \end{aligned}$$

This method works with a system with time delay (see Example 14.10 in your textbook). Routh's method does not.

# Root Locus Diagram

Main Idea: Shows how the closed-loop poles move with different controller gain values.

