Lecture 7
Closed-Loop Analysis
Part I: Laplace Domain Analysis

CHE4400

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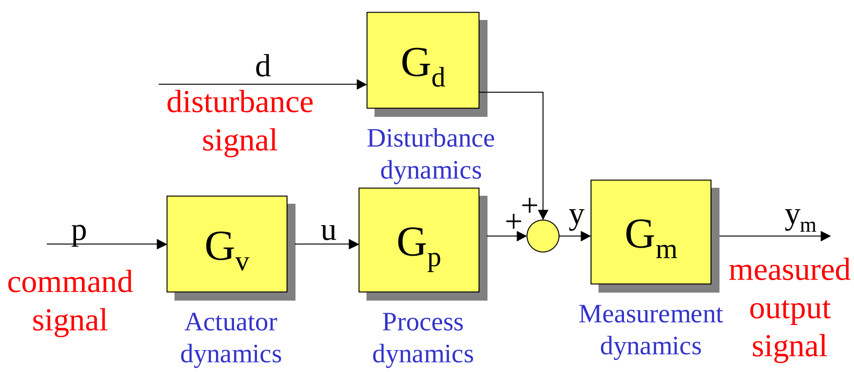
Overview



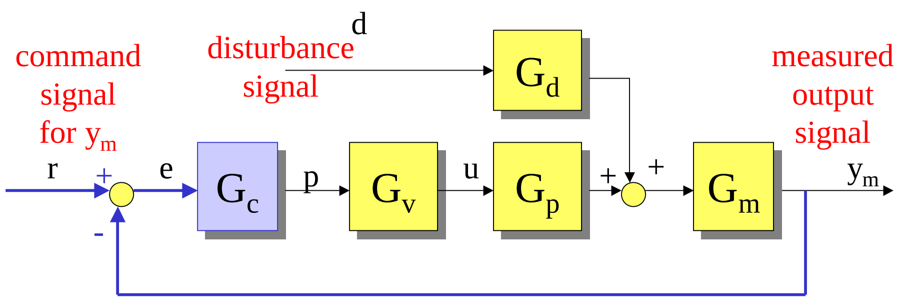
$$\frac{\hat{y}_m(s)}{\hat{p}(s)} = G_m G_p G_v$$

- The gain may not be 1
- The dynamics may be too slow, too oscillatory or unstable.

$$\frac{\hat{y}_m(s)}{\hat{d}(s)} = G_m G_d$$

- The gain may not be 0
- The dynamics may be too slow…….

Overview



$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{G_m G_p G_v G_c}{1+G_m G_p G_v G_c} = \frac{GG_c}{1+GG_c}$$

$$\frac{\hat{y}_m(s)}{\hat{d}(s)} = \frac{G_m G_d}{1+G_m G_p G_v G_c} = \frac{G_{md}}{1+GG_c}$$

(G = $G_m G_p G_v$; G_{md} = $G_m G_d$)

- Different from open-loop!
- Depends on G_c

Analysis and Design Problems

- Analysis: Given particular G, G_{md} and G_c
  - Are the closed-loop dynamics stable?
  - Speed of response? Damping?
  - Gains for y_m/r and y_m/d

Locations of poles and zeros of the closed-loop transfer functions

- Design: Given particular G and G_{md}, choose ("design") G_c so that
  - the closed-loop dynamics are stable.
  - y_m/r has a gain of ~1 and y_m/d has a gain of ~0.
  - the dynamics are sufficiently fast (but not too fast) and smooth (without excessive oscillation).

Model Used for Analysis and Design

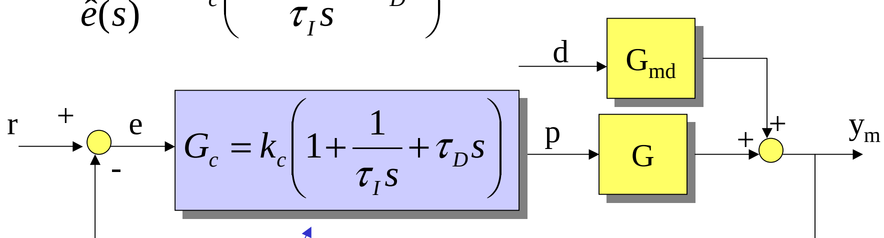
- Case I (Less Frequent)
  - From a fundamental model, perform linearization and Laplace transform of the linearized ODEs to find $G_p(s)$ and $G_d(s)$
  - Find actuator and measurement dynamics G_v and G_m to obtain $G = G_m G_p G_v$ and $G_{md} = G_m G_d$
- Case II (More Frequent)
  - The composite model G is fitted to data of y_m obtained by perturbing p (e.g., by making a step change).
  - G_{md} is often assumed to be same as G (as in IAE/ISE/ITAE tuning rules).

PID Controller

$$p(t) = \bar{p} + k_c\left(e(t) + \frac{1}{\tau_I}\int_0^t e(t')dt' + \tau_D \frac{de}{dt} \right) \quad \Rightarrow$$

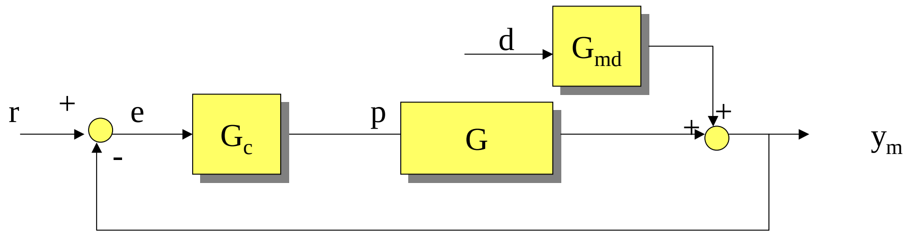
$$\hat{p}'(s) = k_c\left(\hat{e}(s) + \frac{1}{\tau_I s}\hat{e}(s) + \tau_D s\hat{e}(s) \right) \quad \Rightarrow$$

$$\frac{\hat{p}'(s)}{\hat{e}(s)} = k_c\left(1 + \frac{1}{\tau_I s} + \tau_D s \right)$$



However, Gc can be chosen as any transfer function in general.

Calculation of Closed-Loop Functions



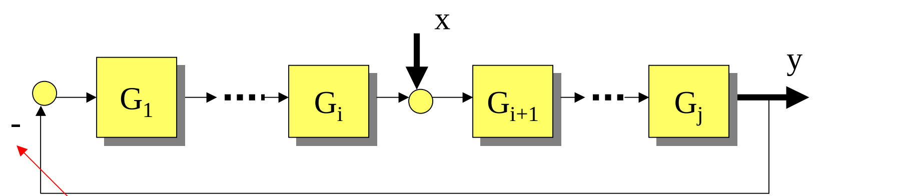
$$\hat{y}_m = G_{md}\hat{d} + G\hat{p} \text{ and } \hat{p} = G_c(\hat{r} - \hat{y}_m) \quad \Rightarrow$$

$$\hat{y}_m = G_{md}\hat{d} + GG_c(\hat{r} - \hat{y}_m) \Rightarrow (1 + GG_c)\hat{y}_m = G_{md}\hat{d} + GG_c\hat{r}$$

$$\frac{\hat{y}_m(s)}{\hat{d}(s)} = \frac{G_{md}}{(1+GG_c)} \text{ and } \frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{GG_c}{(1+GG_c)}$$

See the convenience the Laplace transform affords us?

Calculation of Closed-Loop Functions - General



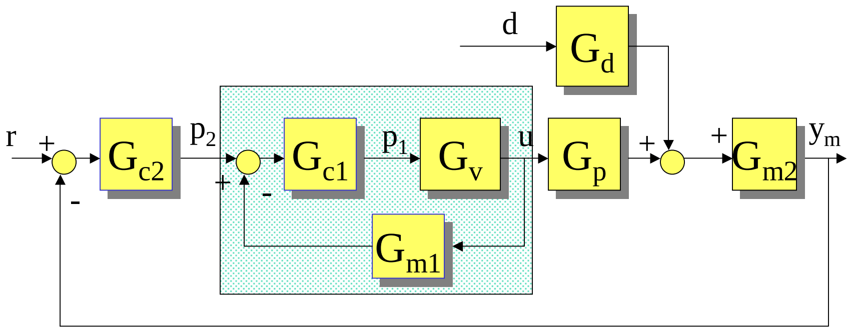
Assume "negative feedback"

open-loop path from x to y

$$\frac{\hat{y}(s)}{\hat{x}(s)} = \frac{G_{i+1}G_{i+2}\cdots G_j}{1 + G_1G_2\cdots G_j} = \frac{G_{OL}}{1 + G_{CL}}$$

Product of all blocks inside the loop

Handling of Cascaded Loops

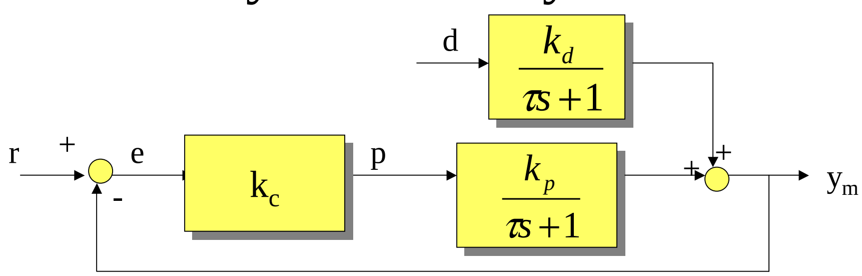


$$\frac{\hat{u}(s)}{\hat{p}_2(s)} = \frac{G_{c1}G_v}{1+G_{c1}G_vG_{m1}} = G_{v\,new}$$

New actuator dynamics modified by the inner loop!

$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{G_{m2}G_pG_{v\,new}G_{c2}}{1+G_{m2}G_pG_{v\,new}G_{c2}}\,;\; \frac{y_m(s)}{d(s)} = \frac{G_{m2}G_d}{1+G_{m2}G_pG_{v\,new}G_{c2}}$$

Analysis of P-only Control



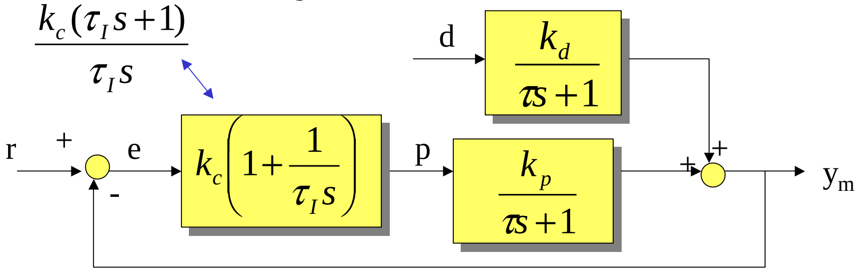
$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \frac{\frac{k_ck_p}{\tau s+1}}{1+\frac{k_ck_p}{\tau s+1}} = \frac{k_ck_p}{\tau s+1+k_ck_p} = \frac{\frac{k_ck_p}{1+k_ck_p}}{\frac{\tau}{1+k_ck_p}s+1}$$

$$\frac{\hat{y}_m(s)}{\hat{d}(s)} = \frac{\frac{k_d}{\tau s+1}}{1+\frac{k_ck_p}{\tau s+1}} = \frac{k_d}{\tau s+1+k_ck_p} = \frac{\frac{k_d}{1+k_ck_p}}{\frac{\tau}{1+k_ck_p}s+1}$$

- Gain is not 1 unless $k_c=\infty$
- Time constant decreases with increasing k_c
- Gain is not 0 unless $k_c=\infty$

Analysis of PI Control

$$\frac{k_c(\tau_I s+1)}{\tau_I s}$$



$$\frac{\hat{y}_m(s)}{\hat{r}(s)}=\frac{\dfrac{k_c k_p(\tau_I s+1)}{\tau_I s(\tau s+1)}}{1+\dfrac{k_c k_p(\tau_I s+1)}{\tau_I s(\tau s+1)}}=\frac{k_c k_p(\tau_I s+1)}{\tau_I s(\tau s+1)+k_c k_p(\tau_I s+1)}$$

$$=\frac{(\tau_I s+1)}{\dfrac{\tau_I \tau}{k_c k_p}s^2+\dfrac{1+k_c k_p}{k_c k_p}\tau_I s+1}$$

- Gain =1 always! No offset.
- 2nd order dynamics
- Underdamped dynamics for very small τ_I

Characteristic Equation

$$1+GG_c=0$$

- Roots of the above equation are the <u>poles of the closed-loop functions</u> (important information for analyzing closed-loop dynamics)
- For stability, make sure all the roots are in the Left-Half-Plane (negative real parts)
  - Can be checked by Routh's test (14.6.3 of the textbook)
  - Or by Direct Substitution (14.6.4)

Example: Routh's Test

Main Idea: Form a Routh array to see if any roots are in the RHP.

$$1+\frac{6K_c}{(2s+1)(4s+1)(6s+1)}=0 \quad \Rightarrow$$

$$48s^3+44s^2+12s+(1+6K_c)=0$$

$$\begin{array}{ll} 48 & 12 \\ 44 & 1+6K_c \\ \hline \frac{120}{11}-\frac{72}{11}K_c & 0 \\ 1+6K_c & \end{array}$$

$$\frac{44\times 12-48(1+6Kc)}{44}$$

$$\frac{\left(\frac{120}{11}-\frac{72}{11}K_c\right)\times(1+6K_c)-44\times 0}{\left(\frac{120}{11}-\frac{72}{11}K_c\right)}$$

Must be all positive for closed-loop stability!

$$\frac{120}{11}-\frac{72}{11}K_c>0 \;\Rightarrow\; K_c<\frac{5}{3}$$

$$1+6K_c>0 \;\Rightarrow\; K_c>-\frac{1}{6}$$

Example: Direct Substitution

Main Idea: At the limits of instability, the closed-loop poles will be on the imaginary axis (between LHP and RHP)

$$48s^3+44s^2+12s+(1+6K_c)=0 \xrightarrow{s=j\omega}$$

$$-48j\omega^3-44\omega^2+12j\omega+(1+6K_c)=0 \quad \rightarrow$$

$$(-48\omega^3+12\omega)j+\left[-44\omega^2+(1+6K_c)\right]= \quad \rightarrow$$

$$\begin{array}{l} -48\omega^3+12\omega=0 \\ -44\omega^2+(1+6K_c)=0 \end{array} \rightarrow \begin{array}{l} \omega=0,K_c=-1/6 \\ \omega=\pm 1/2,K_c=5/3 \end{array}$$

This method works with a system with time delay (see Example 14.10 in your textbook). Routh's method does not.

Root Locus Diagram

Main Idea: Shows how the closed-loop poles move with different controller gain values.