

Lecture 3
Dynamic Modeling
Part III: Simulation and Analysis

CHE4400

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Standard Form of the Model

- 2 dependent variables and two independent variables:

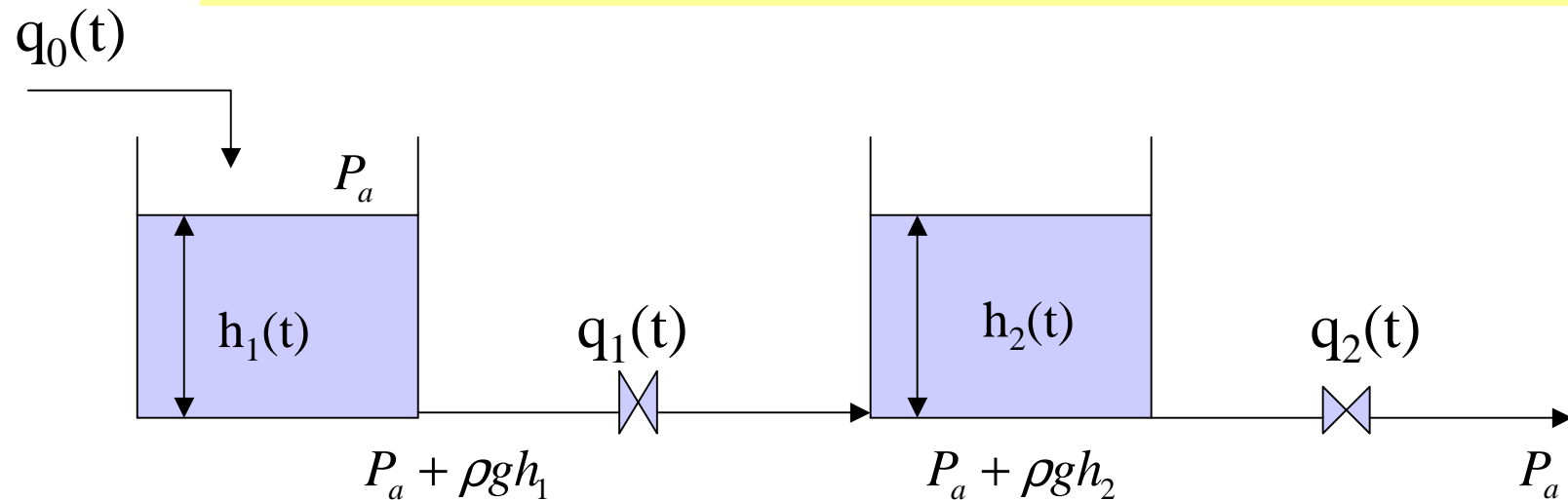
$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, u_1, u_2) \\ \dot{x}_2 &= f_2(x_1, x_2, u_1, u_2)\end{aligned}$$

f_1, f_2 : some (nonlinear) functions of x_1, x_2, u_1, u_2

- There can be many more dependent variables and many more independent variables. **What would be the form with n dependent variables and m independent variables?**

Example: Interacting Tanks

$$q_1 = C_{v1} \sqrt{(P_a + \rho g h_1) - (P_a + \rho g h_2)} = C_{v1} \sqrt{\rho g (h_1 - h_2)}$$



$$q_2 = C_{v2} \sqrt{P_a + \rho g h_2 - P_a} = C_{v2} \sqrt{\rho g h_2}$$

Standard Form

$$\frac{d(A_1 h_1 \rho)}{dt} = \rho q_0 - \underbrace{\rho C_{v1} \sqrt{\rho g (h_1 - h_2)}}_{q_1}$$

$$\frac{d(A_2 h_2 \rho)}{dt} = \underbrace{\rho C_{v1} \sqrt{\rho g (h_1 - h_2)}}_{q_1} - \underbrace{\rho C_{v2} \sqrt{\rho g h_2}}_{q_2}$$



where $x_1 \equiv h_1$, $x_2 \equiv h_2$, $u_1 \equiv q_0$

$$\underbrace{\frac{dh_1}{dt}}_{\dot{x}_1} = \frac{q_0 - C_{v1} \sqrt{\rho g (h_1 - h_2)}}{A_1}$$

$f_1(x_1, x_2, u_1)$

$$\underbrace{\frac{dh_2}{dt}}_{\dot{x}_2} = \frac{C_{v1} \sqrt{\rho g (h_1 - h_2)} - C_{v2} \sqrt{\rho g h_2}}{A_2}$$

$f_2(x_1, x_2, u_1)$

What can you do with the model?

- **Numerical integration** (“simulation”) to investigate the time behavior of the dependent variables to a particular $x(0)$ (**initial condition**) and $u(t)$, $t \geq 0$ (**independent variable**).

What does “numerically solving” the ODEs mean?

- Given: $x(0)$ and $(u(t), t \geq 0)$
- Obtain $(x(t), t > 0)$
- **Analysis:**
 - Linearization
 - Analytical solution via Laplace Transform

Numerical Integration

- Start with $x_1(0)$ and $x_2(0)$. Set $t=0$.
- Take an incremental step (of size Δt , **which cannot be large, why?**) forward in time by solving

Forward Euler: *Explicit Integration*

$$x_1(t + \Delta t) = x_1(t) + \Delta t \cdot f_1(x_1(t), \dots)$$

$$x_2(t + \Delta t) = x_2(t) + \Delta t \cdot f_2(x_1(t), \dots)$$

Backward Euler: *Implicit Integration*

$$x_1(t + \Delta t) = x_1(t) + \Delta t \cdot f_1(x_1(t + \Delta t), \dots)$$

$$x_2(t + \Delta t) = x_2(t) + \Delta t \cdot f_2(x_1(t + \Delta t), \dots)$$

Trapezoidal (2nd order R-K): *Implicit Integration*

$$x_1(t + \Delta t) = x_1(t) + \Delta t \frac{f_1(x_1(t), \dots) + f_1(x_1(t + \Delta t), \dots)}{2}$$

$$x_2(t + \Delta t) = x_2(t) + \Delta t \frac{f_2(x_1(t), \dots) + f_2(x_1(t + \Delta t), \dots)}{2}$$

- Repeat this until you reach the desired end time

Equilibrium Calculation

- At steady state, $d/dt=0$

$$0 = f_1(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)$$

$$0 = f_2(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)$$

- Given the steady state values of the independent variables, one can calculate the corresponding steady-state values of the dependent values by solving the above equation.

Solving Algebraic Equations Numerically

- Newton Iteration

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} f_1(\bar{x}_1, \bar{x}_2) \\ f_2(\bar{x}_1, \bar{x}_2) \end{bmatrix} \\ &\approx \begin{bmatrix} f_1(\bar{x}_1^i, \bar{x}_2^i) \\ f_2(\bar{x}_1^i, \bar{x}_2^i) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1^i, \bar{x}_2^i)} \begin{bmatrix} \bar{x}_1 - \bar{x}_1^i \\ \bar{x}_2 - \bar{x}_2^i \end{bmatrix} \end{aligned}$$

current guess of the solutions


By solving the above approximate equation, one gets iterative formula:

$$\begin{bmatrix} \bar{x}_1^{i+1} \\ \bar{x}_2^{i+1} \end{bmatrix} = \begin{bmatrix} \bar{x}_1^i \\ \bar{x}_2^i \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1^i, \bar{x}_2^i)}^{-1} \begin{bmatrix} f_1(\bar{x}_1^i, \bar{x}_2^i) \\ f_2(\bar{x}_1^i, \bar{x}_2^i) \end{bmatrix}$$

Example

$$0 = \bar{q}_0 - C_{v1} \sqrt{\rho g (\bar{h}_1 - \bar{h}_2)}$$

$$0 = C_{v1} \sqrt{\rho g (\bar{h}_1 - \bar{h}_2)} - C_{v2} \sqrt{\rho g \bar{h}_2}$$


 \bar{h}_1^0, \bar{h}_2^0 ← Initial guess, $i=0$
 Newton iteration (repeat until convergence!)

$$\begin{bmatrix} \bar{h}_1^{i+1} \\ \bar{h}_2^{i+1} \end{bmatrix} = \begin{bmatrix} \bar{h}_1^i \\ \bar{h}_2^i \end{bmatrix} - M_i^{-1} \begin{bmatrix} \bar{q}_0 - C_{v1} \sqrt{\rho g (\bar{h}_1^i - \bar{h}_2^i)} \\ C_{v1} (\bar{h}_1^i - \bar{h}_2^i) - C_{v2} \sqrt{\rho g \bar{h}_2^i} \end{bmatrix}$$

$$M_i = \begin{bmatrix} -\frac{C_{v1} \sqrt{\rho g}}{2\sqrt{(\bar{h}_1^i - \bar{h}_2^i)}} & \frac{C_{v1} \sqrt{\rho g}}{2\sqrt{(\bar{h}_1^i - \bar{h}_2^i)}} \\ \frac{C_{v1} \sqrt{\rho g}}{2\sqrt{(\bar{h}_1^i - \bar{h}_2^i)}} & -\frac{C_{v1} \sqrt{\rho g}}{2\sqrt{(\bar{h}_1^i - \bar{h}_2^i)}} - \frac{C_{v2} \sqrt{\rho g}}{2\sqrt{\bar{h}_2^i}} \end{bmatrix}$$

Linearization (1st Order Approximation of ODEs Around an Equilibrium)

$$\dot{x}_1 = f_1(x_1, x_2, u_1, u_2)$$

$$\dot{x}_2 = f_2(x_1, x_2, u_1, u_2)$$



1st-order Taylor series expansion at the equilibrium
 $(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} f_1(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2) \\ f_2(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix} \\ + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}_{(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)} \begin{bmatrix} u_1 - \bar{u}_1 \\ u_2 - \bar{u}_2 \end{bmatrix}$$

Standard Form of Linearized Model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix} \\ + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}_{(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)} \begin{bmatrix} u_1 - \bar{u}_1 \\ u_2 - \bar{u}_2 \end{bmatrix}$$



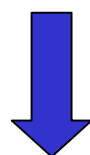
$$x_1' \equiv x_1 - \bar{x}_1, \text{ etc.}$$

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} = A \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + B \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$$

Example

$$\underbrace{\frac{dh_1}{dt}}_{\dot{x}_1} = \underbrace{\frac{q_0 - C_{v1}\sqrt{\rho g(h_1 - h_2)}}{A_1}}_{f_1(x_1, x_2, u_1)}$$

$$\underbrace{\frac{dh_2}{dt}}_{\dot{x}_2} = \underbrace{\frac{C_{v1}\sqrt{\rho g(h_1 - h_2)} - C_{v2}\sqrt{\rho g h_2}}{A_2}}_{f_2(x_1, x_2, u_1)}$$



Linearization at $\bar{h}_1, \bar{h}_2, \bar{q}_0$

$$\begin{bmatrix} \frac{dh_1'}{dt} \\ \frac{dh_2'}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{C_{v1}\sqrt{\rho g}}{2A_1\sqrt{(\bar{h}_1 - \bar{h}_2)}} & \frac{C_{v1}\sqrt{\rho g}}{2A_1\sqrt{(\bar{h}_1 - \bar{h}_2)}} \\ \frac{C_{v1}\sqrt{\rho g}}{2A_2\sqrt{(\bar{h}_1 - \bar{h}_2)}} & -\frac{C_{v1}\sqrt{\rho g}}{2A_2\sqrt{(\bar{h}_1 - \bar{h}_2)}} - \frac{C_{v2}\sqrt{\rho g}}{2A_2\sqrt{\bar{h}_2}} \end{bmatrix} \begin{bmatrix} h_1' \\ h_2' \end{bmatrix} + \begin{bmatrix} 1 \\ A_1 \\ 0 \end{bmatrix} \dot{q}_0$$

Laplace Transform – Main Idea

Linear ODEs

Example : 1st order ODE

$$\tau \frac{dy}{dt} + y(t) = u(t)$$

$u(t)$ is a given time function.

“solve”
involves
integration.



Solution

$$y(t) = \exp\left(-\frac{t}{\tau}\right)y(0) + \int_0^t \exp\left(-\frac{(t-\zeta)}{\tau}\right)u(\zeta) d\zeta$$

Table,
Memorization

**Laplace
Transform**



$$(\tau s + 1)\hat{y}(s) = \hat{u}(s)$$

$\hat{u}(s)$ is a function of s ,
Laplace transform of $u(t)$.

“solve” here
involves
just algebra!



**Inverse
Laplace
Transform**



Partial Fraction E.,
Table,
Memorization

$$\hat{y}(s) = \frac{1}{(\tau s + 1)}\hat{u}(s)$$

Laplace Transform – Key Points(1)

- Formula $\hat{f}(s) \equiv L\{f(t)\} \equiv \int_0^{\infty} f(t)e^{-st} dt \Rightarrow L^{-1}\{\hat{f}(s)\} = f(t)$

- Laplace Transform for simple signals

- Steps, ramps, exponential decay or rise, pulse, impulse, etc.
- Can be found by evaluating the integral.
- See Table C2 of your textbook
- Must memorize the simple ones.

This is what enables us to convert linear ODEs into linear algebraic equations

- Important Properties

$$L\left\{\frac{df}{dt}\right\} \equiv s \cdot \hat{f}(s) - f(0), \quad L\left\{\frac{d^2 f}{dt^2}\right\} \equiv s^2 \cdot \hat{f}(s) - s \cdot f(0) - f'(0), \text{ etc.}$$

$$L\left\{\int_0^t f(\zeta)d\zeta\right\} = \frac{1}{s} \hat{f}(s); \quad L\{f(t - \delta)\} = \hat{f}(s)e^{-\delta s}$$

$\hat{f}(s)$	$f(t)$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$s^{-3/2}$	$2\sqrt{\frac{t}{\pi}}$
$\frac{\Gamma(k)}{s^k} (k \geq 0)$	t^{k-1}
$\frac{\Gamma(k)}{(s+a)^k} (k \geq 0)$	$t^{k-1}e^{-at}$
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{\tau s + 1}$	$\frac{1}{\tau} e^{-t/\tau}$
$\frac{1}{s(\tau s + 1)}$	$1 - e^{-t/\tau}$
$\frac{1}{(s+a)^2}$	te^{-at}
$\frac{1}{(\tau s + 1)^2}$	$\frac{t}{\tau^2} e^{-t/\tau}$
$\frac{1}{s(\tau s + 1)^2}$	$1 - (1 + t/\tau)e^{-t/\tau}$
$\frac{1}{(s+a)^n} (n = 1, 2, \dots)$	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$
$\frac{1}{(\tau s + 1)^n} (n = 1, 2, \dots)$	$\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$
$\frac{1}{(s+a)(s+b)}$	$\frac{1}{(b-a)} (e^{-at} - e^{-bt})$

$\hat{f}(s)$	$f(t)$
$\frac{s}{(s+a)(s+b)}$	$\frac{1}{(b-a)}(be^{-bt} - ae^{-at})$
$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab} \left[1 + \frac{1}{(a-b)}(be^{-at} - ae^{-bt}) \right]$
$\frac{b}{s^2 + b^2}$	$\sin bt$
$\frac{b}{s^2 - b^2}$	$\sinh bt$
$\frac{s}{s^2 + b^2}$	$\cos bt$
$\frac{s}{s^2 - b^2}$	$\cosh bt$
$\frac{b^2}{s(s^2 + b^2)}$	$(1 - \cos bt)$
$\frac{b}{(s+a)^2 + b^2}$	$e^{-at} \sin bt$
$\frac{s+a}{(s+a)^2 + b^2}$	$e^{-at} \cos bt$
$\hat{f}(s+a)$	$e^{-at} f(t)$
$e^{-bs} \hat{f}(s)$	$f(t-b)$; with $f(t) = 0$ for $t < 0$

Laplace Transform – Key Points(2)

- Laplace Transform is a linear operation.

$$L\{a \cdot f_1(t) + b \cdot f_2(t)\} = a \cdot \hat{f}_1(s) + b \cdot \hat{f}_2(s)$$

- Inverse Laplace Transform

- Needed to take the solution obtained through Laplace transform back to the time domain.
- The formula involves complex contour integral.
- Use partial fraction expansion to break the solution down to small pieces and use the table (or your memory) to invert.
- Final and Initial Value Theorem.

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \cdot \hat{f}(s); \quad f(0) = \lim_{s \rightarrow \infty} s \cdot \hat{f}(s)$$

Laplace Transform – Key Points(3)

- Laplace transform does not work for solving nonlinear ODEs.

$$L\{y^2(t)\} \neq \hat{y}^2(s); L\{y(t)u(t)\} \neq \hat{y}(s) \cdot \hat{u}(s)$$

- See the supplementary note (posted on the class homepage) on Laplace transform for details.

Transfer Function

- Linear differential equation with a general forcing function (input)

$$a \frac{dy}{dt} + y(t) = ku(t)$$

- 1st Order

- 2nd Order

- Etc.

$$b \frac{d^2 y}{dt^2} + a \frac{dy}{dt} + y(t) = ku(t)$$

- We can solve the eqn. for a specific forcing function but we can also leave it general and take Laplace transform (with the initial condition of zero) to arrive at a **general relationship between output and input.**

$$\hat{y}(s) = \frac{k}{(as+1)} \hat{u}(s); \quad \hat{y}(s) = \frac{k}{(bs^2 + as + 1)} \hat{u}(s)$$

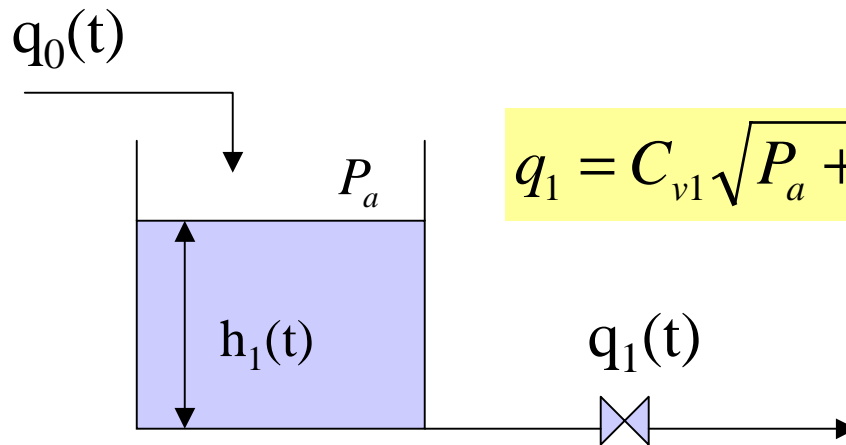
- With the transfer function, one can conveniently calculate the response of the output to *any* input by multiplication.

$$\hat{y}(s) = G(s) \cdot \hat{u}(s)$$

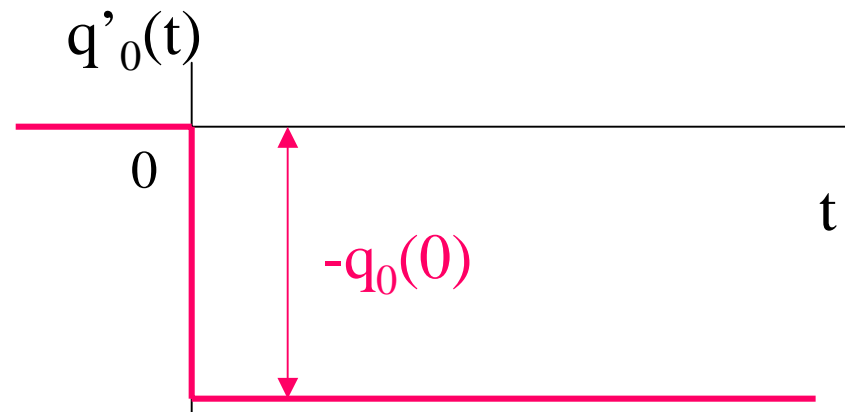
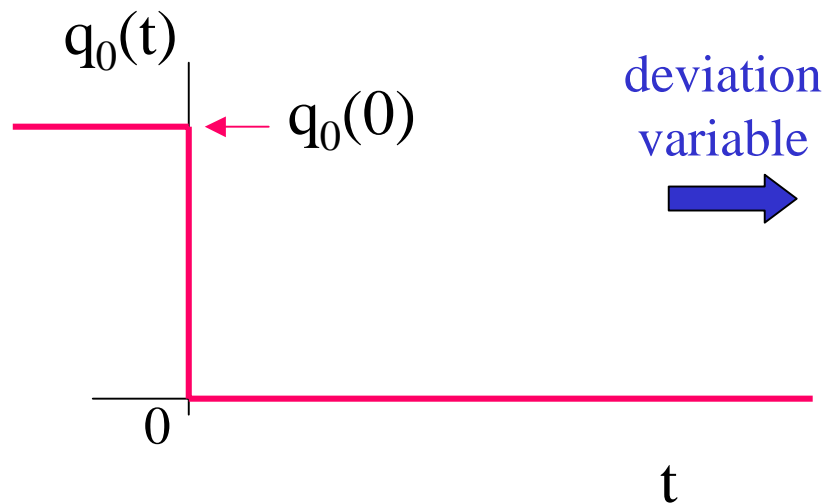
Example: Single-Tank Draining

Assume that you start at the steady state. ($q_0(0) = \bar{q}$; $h_1(0) = \bar{h}_1$)

$$\left(= \frac{\bar{q}_0^2}{C_{v1}^2 \rho g} \right)$$



$$q_1 = C_{v1} \sqrt{P_a + \rho g h_1 - P_a} = C_{v1} \sqrt{\rho g h_1}$$



Mass Balances

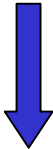
$$\frac{d(A_1 h_1 \rho)}{dt} = \rho q_0 - \underbrace{\rho C_{v1} \sqrt{\rho g h_1}}_{q_1}$$



$$A_1 \frac{dh_1}{dt} = q_0 - C_{v1} \sqrt{\rho g h_1}$$

Linearization (Linear Approximation)

$$A_1 \frac{dh_1}{dt} = q_0 - C_{v1} \sqrt{\rho g h_1}$$

$q_0' \equiv q_0 - \bar{q}_0, h_1' \equiv h_1 - \bar{h}_1, \text{etc.}$  Linearize at the steady state of $q_0 = \bar{q}_0, h_1 = \bar{h}_1$

$$A_1 \frac{dh_1'}{dt} = q_0' - \underbrace{\left(\frac{C_{v1} \sqrt{\rho g}}{2\sqrt{\bar{h}_1}} \right)}_{1/R_1} h_1'$$

The above can be solved (using Laplace Transform) but will the linear model be valid throughout the entire draining experiment?

Solution Based on the Linearized Model

$$A_1 \frac{dh_1'}{dt} = q_0' - \frac{1}{R_1} h_1' \text{ with } h_1'(0) = 0 \text{ and } q_0'(s) = -\frac{\bar{q}_0}{s}$$

↓ L

$$\left(A_1 s + \frac{1}{R_1} \right) \hat{h}_1'(s) = -\frac{\bar{q}_0}{s} \Rightarrow \hat{h}_1'(s) = \frac{-\bar{q}_0}{s \left(A_1 s + \frac{1}{R_1} \right)} = \frac{-\bar{q}_0 R_1}{s} + \frac{\bar{q}_0 R_1}{\left(s + \frac{1}{A_1 R_1} \right)}$$

↓ L^{-1}

$$h_1'(t) = -\bar{q}_0 R_1 \left(1 - \exp\left(-\frac{t}{A_1 R_1} \right) \right) \Rightarrow h_1(t) = \bar{h}_1 - \bar{q}_0 R_1 \left(1 - \exp\left(-\frac{t}{A_1 R_1} \right) \right)$$

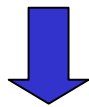
- We can see that $h_1(t)$ is an exponentially decaying function of t .

Real Solution

Integrate the differential equation from time 0 to t to obtain

$$A_1 \frac{dh_1}{dt} = -C_{v1} \sqrt{\rho g h_1}, \quad h(0) = \bar{h}$$

$$\frac{1}{\sqrt{h_1}} \frac{dh_1}{dt} = -\frac{C_{v1} \sqrt{\rho g}}{A_1},$$



$$2\left(\sqrt{h_1(t)} - \sqrt{\bar{h}_1}\right) = \frac{-C_{v1} \sqrt{\rho g}}{A_1} (t - 0) \Rightarrow h_1(t) = \left(\frac{-C_{v1} \sqrt{\rho g}}{2A_1} t + \sqrt{\bar{h}_1}\right)^2$$

We can see that $h_1(t)$ is a quadratic function of t.

Laplace Transform of a 2x2 System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



Laplace Transform

$$\begin{bmatrix} s\hat{x}_1(s) - x_1(0) \\ s\hat{x}_2(s) - x_2(0) \end{bmatrix} = A \begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \end{bmatrix} + B \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \end{bmatrix}$$

$$\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - A \right) \begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + B \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \end{bmatrix} = \underbrace{\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - A \right)^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\text{Effect of Non-Zero Initial Condition}} + \underbrace{\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - A \right)^{-1} B \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \end{bmatrix}}_{\text{Effect of Forcing Function}}$$

Effect of Non-Zero Initial Condition

Effect of Forcing Function

Example – Interacting Tanks

$$\begin{bmatrix} \frac{dh_1'}{dt} \\ \frac{dh_2'}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{A_1 R_1} & \frac{1}{A_1 R_1} \\ \frac{1}{A_2 R_1} & -\frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \end{bmatrix} \begin{bmatrix} h_1' \\ h_2' \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} q_0'$$



Laplace Transform

$$\begin{bmatrix} \hat{h}_1'(s) \\ \hat{h}_2'(s) \end{bmatrix} = \underbrace{\begin{bmatrix} s + \frac{1}{A_1 R_1} & -\frac{1}{A_1 R_1} \\ -\frac{1}{A_2 R_1} & s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \end{bmatrix}^{-1}}_{\text{Effect of Non-Zero Initial Condition}} \begin{bmatrix} h_1'(0) \\ h_2'(0) \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} s + \frac{1}{A_1 R_1} & -\frac{1}{A_1 R_1} \\ -\frac{1}{A_2 R_1} & s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \end{bmatrix}^{-1}}_{\text{Effect of Inlet Flowrate Change}} \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} \hat{q}_0'(s)$$

Effect of Inlet Flowrate Change

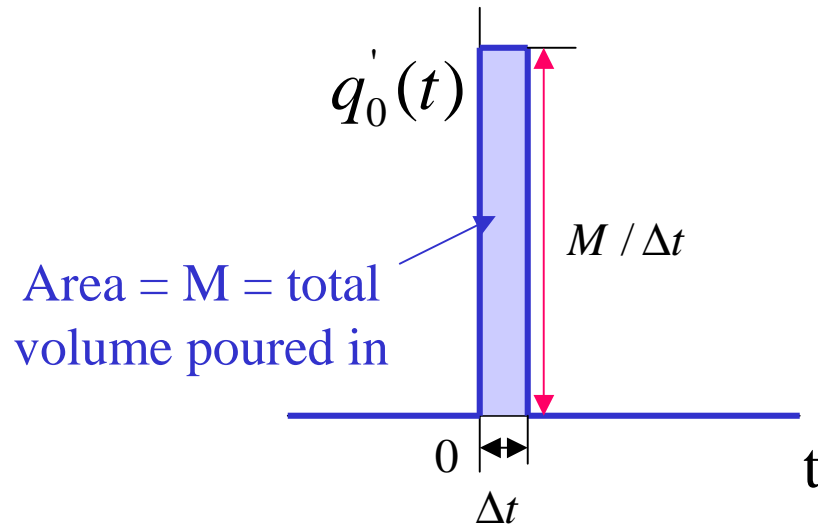
Example

- How will h_1 and h_2 respond in time if you “quickly” poured a beaker of water (M ml) into the first tank? Assume that you start at the equilibrium. $\Rightarrow h_1'(0) = h_2'(0) = 0$

$$\begin{bmatrix} \hat{h}_1'(s) \\ \hat{h}_2'(s) \end{bmatrix} = \begin{bmatrix} \frac{\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right) \frac{1}{A_1}}{\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right) \left(s + \frac{1}{A_1 R_1} \right) - \frac{1}{A_1 A_2 R_1^2}} \hat{q}_0'(s) \\ \frac{1}{A_1 A_2 R_1} \hat{q}_0'(s) \\ \frac{\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right) \left(s + \frac{1}{A_1 R_1} \right) - \frac{1}{A_1 A_2 R_1^2}}{\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right) \left(s + \frac{1}{A_1 R_1} \right) - \frac{1}{A_1 A_2 R_1^2}} \hat{q}_0'(s) \end{bmatrix}$$

Transfer
Function

Example



$\Delta t \rightarrow 0$

$$\hat{q}_0'(s) = M$$

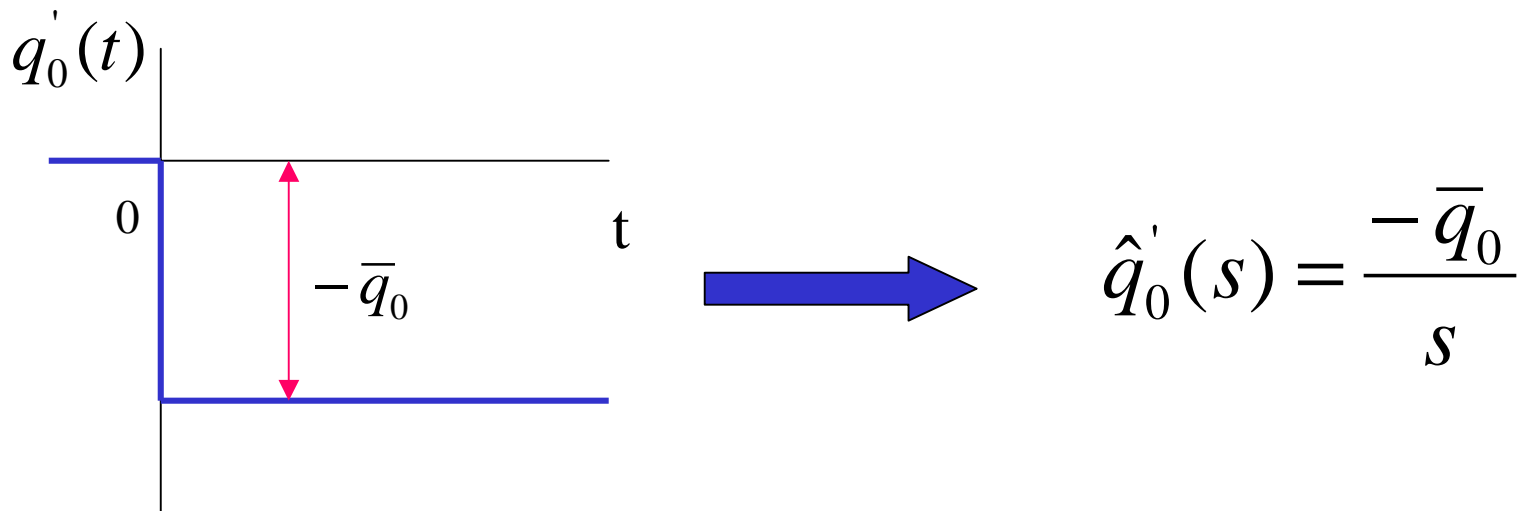
$$\begin{bmatrix} \hat{h}_1'(s) \\ \hat{h}_2'(s) \end{bmatrix} = \frac{\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right) \frac{M}{A_1}}{\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right) \left(s + \frac{1}{A_1 R_1} \right) - \frac{1}{A_1 A_2 R_1^2}} \frac{M}{A_1 A_2 R_1}$$

Inverse Laplace Transform

$$\begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} = \begin{bmatrix} h_1'(t) \\ h_2'(t) \end{bmatrix} + \begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix}$$

Example

- How will h_1 and h_2 respond in time if you shut off the inlet flow at $t=0$? Assume that you start at the equilibrium. $\Rightarrow h_1'(0) = h_2'(0) = 0$

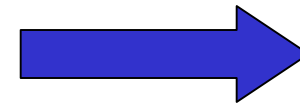


Note for the lab: Since h gets far away from the steady-state value, the linearized model may become less and less valid as time increases. Use the nonlinear model directly in this case.

Example

$$\begin{bmatrix} \hat{h}_1'(s) \\ \hat{h}_2'(s) \end{bmatrix} = \begin{bmatrix} \frac{-\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)\right) \bar{q}_0}{A_1 s}}{\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)\right) \left(s + \frac{1}{A_1 R_1}\right) - \frac{1}{A_1 A_2 R_1^2}} \\ -\frac{1}{A_1 A_2 R_1} \frac{\bar{q}_0}{s}}{\left(s + \frac{1}{A_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)\right) \left(s + \frac{1}{A_1 R_1}\right) - \frac{1}{A_1 A_2 R_1^2}} \end{bmatrix}$$

*Inverse
Laplace
Transform*



$$\begin{bmatrix} h_1'(t) \\ h_2'(t) \end{bmatrix}$$



$$\begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} = \begin{bmatrix} h_1'(t) \\ h_2'(t) \end{bmatrix} + \begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix}$$

Caution: The Linearized Model may not be valid for the entire duration of draining, as we mentioned earlier.

Example: Lab 5 (pH Experiment)

$$\begin{bmatrix} \frac{dC_A}{dt} \\ \frac{dC_B}{dt} \end{bmatrix} = \begin{bmatrix} F_A C_{Ai} - (F_A + F_B) C_A \\ F_B C_{Bi} - (F_A + F_B) C_B \end{bmatrix}$$

f: fraction of acid that reacted with base to produce water

$$pH = -\log_{10}[C_A(1-f)]$$

$$f = \frac{C_A + C_B - \sqrt{(C_A - C_B)^2 + 4 \cdot 10^{-14}}}{2C_A} \Leftrightarrow [C_B - f \cdot C_A][C_A - f \cdot C_A] = 10^{-14}$$

Linearization w.r.t. $\bar{C}_A, \bar{C}_B, \bar{F}_A, \bar{F}_B$ should give

$$\begin{bmatrix} \frac{dC'_A}{dt} \\ \frac{dC'_B}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C'_A \\ C'_B \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} F'_A \\ F'_B \end{bmatrix}$$

$$pH' = [c_1 \quad c_2] \begin{bmatrix} C'_A \\ C'_B \end{bmatrix}$$

Q: Where are the nonlinearities in the equations?

Generalization (n x n System)

Matrix Differential Equation

$$\dot{x}' = A x' + B u'$$

$$y' = Cx' + D u'$$

$$y'(t) = Cx'(t) + D u'(t)$$

y is a particular linear combination of x and u that you want to calculate.

$$L \quad \downarrow \quad \uparrow \quad L^{-1}$$

$$\hat{x}'(s) = (sI - A)^{-1} B \hat{u}'(s)$$

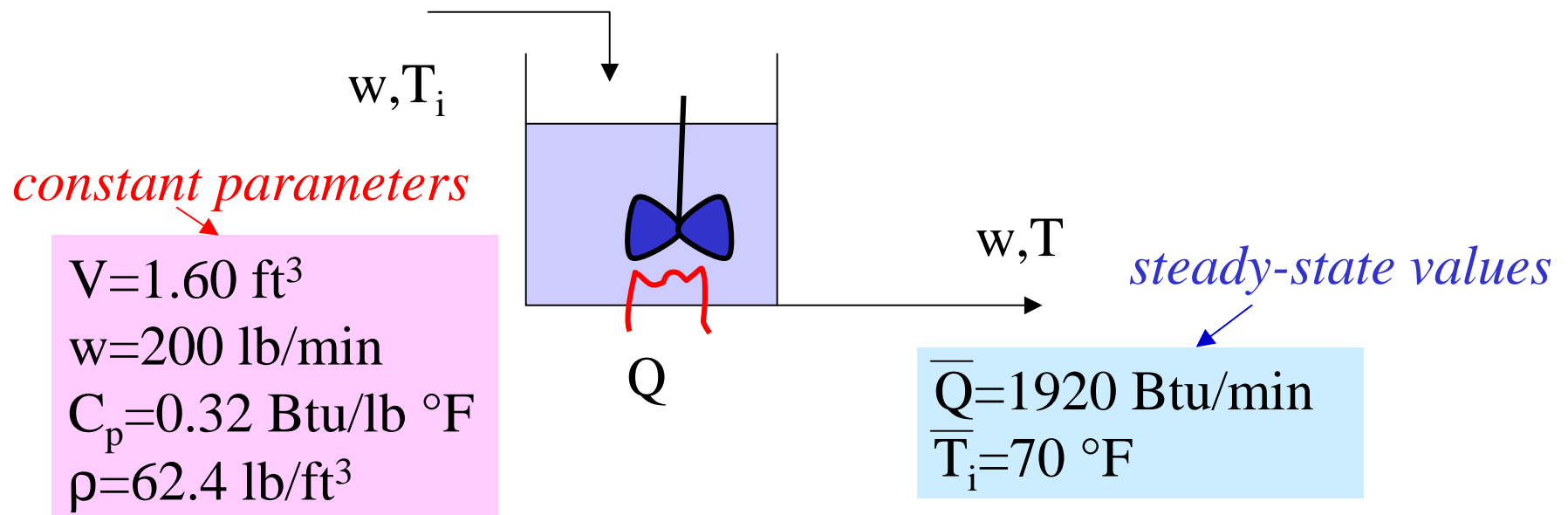
$$\hat{y}'(s) = \underbrace{\left(C(sI - A)^{-1} B + D \right)}_{G(s)} \hat{u}'(s)$$

G(s) ← Transfer function between u and y

Which one would you rather use for calculating x and y for given u?

Sample Test Problem 1

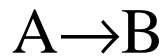
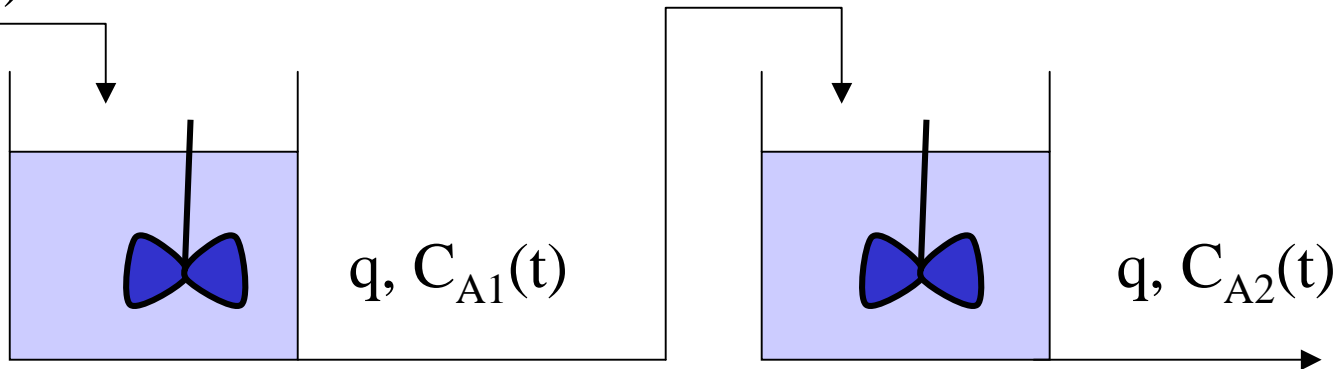
(Solutions are in the supplementary notes)



Suppose that T_i increased from $70 \text{ } ^\circ\text{F}$ to $90 \text{ } ^\circ\text{F}$ and Q dropped from 1920 Btu/min to 1600 Btu/min at $t=0$. Calculate the transient response of T . Assume that the tank was at the steady state at $t=0$.

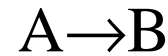
Sample Test Problem 2

$q=2\text{m}^3/\text{min},$
 $C_{A0}(t)$



$r_1 = 0.5C_{A1}(t) / \text{min}$

$V_1 = 4\text{m}^3$



$r_2 = 0.5C_{A2}(t) / \text{min}$

$V_2 = 3\text{m}^3$

At steady state, $C_{A0} = \bar{C}_{A0} = 1\text{kgmol}/\text{m}^3$

At $t=0$, engineer poured 2.5 kgmols of pure A into the first tank. Calculate the response of $C_{A1}(t)$ and $C_{A2}(t)$. Assume that the reactors were at their steady state at $t=0$.